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Prudence and preference for flexibility gain

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Prudence and preference for flexibility gain[*]

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Abstract

Under the expected utility paradigm, prudence (positive third derivative of the utility function: $u''' > 0$) is usually associated with the preference of a risk averse individual for a current certain outcome over a future one, obtained by adding a zero mean lottery to the current outcome. This preference is measured by the utility premium, which is higher for a prudent individual the lower her initial wealth is. However, when the individual has to make a costly investment before attaining the outcome, she may prefer to delay that investment. This translates into a preference for a later over the earlier outcome. Therefore, prudence cannot be associated with the magnitude of the utility premium. In this paper, we show that for an individual who prefers to delay the investment, prudence is actually related to the economic benefit granted by that delay. In particular, if the cost distribution is uniform, a lower expected unit cost of acquiring the good is associated with a greater benefit of the investment delay if and only if $u''' > 0$. We also show that the preference of an individual for facing a distribution with a lower expected unit cost and/or a wider support of the unit cost increases with u'''. We provide two applications of this result looking at a principal-agent relationship and an investment timing problem in production capacity. We also show that, whereas prudence induces a delay of investment in capacity, it has an opposite effect when the investment has the nature of a preventive effort.

Keywords: Prudence; Downside risk aversion; Sequential screening; Real Options

J.E.L. Classification Numbers: D81

1   Introduction

A few different interpretations of prudence have been provided by the literature, all mathematically equivalent to $u''' > 0$, where u is the utility function of some given individual. This notion was introduced by Kimball [18] as a precautionary motive for savings in response to an increased risk about future revenues. A prudent individual saves more and an imprudent individual saves less when faced with an increased risk over her future budget. In another interpretation, strictly related to Kimball's definition, if a risk is added to the future wealth of the individual, the reduction in her expected utility represents a utility premium, which is greater the lower her initial wealth is (Hanson and Menezes [12]).[1] A somehow different interpretation of prudence refers to the preference over specific lotteries. If the prudent individual is asked to choose between two lotteries, which only differ in that some given risk is added to a good outcome in one of the lotteries and to a bad outcome in the other, then the former lottery will be chosen (Menezes et al. [16]). Accordingly, a prudent individual is said to be downside risk averse. Still according to this interpretation, when the individual is called upon to choose between a certain outcome today and an unknown outcome tomorrow, prudence is viewed as representing the link between the utility premium and the initial wealth (see Menezes et al. [16] and Eeckhoudt and Schlesinger [8]).

Broadly speaking, in the interpretations provided by the literature, prudence is related to how much an individual dislikes facing an uncertain future outcome as compared to the current certain one, which is consistent with Kimball's definition. In each of them, one can easily identify the individual as being a consumer who derives some utility from her income (as expressed through the indirect utility function). We can say that prudence expresses the extent to which the consumer dislikes uncertainty being added to that income in the future, provided that preference is not explained by risk aversion, to which prudence is related.

Since the argument of the utility function is money, the utility function $u\left(\cdot\right)$ of an investor is tantamount to that of a consumer. This traces back to Bernoulli's examples, in which the investor derives a utility from money, consistent with Bernoulli's and Cramer's conjecture that the investor cares about the utility derived from money rather than about money *per se*.[2] Hence, for a prudent investor to accept making a risky investment that adds a lottery to her initial wealth, she must obtain a benefit that more than compensates for the utility premium (the cost in terms of expected utility).

What about an investor who has a strict preference for future investment opportunities over the current ones? That investor prefers *later* outcomes to *early* outcomes. Is that preference still related to prudence? To the best of my knowledge, this kind of situations has not been considered in Decision theory hitherto.

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[1] As pointed out by Eeckhoudt and Schlesinger [8], the notion of utility premium seems to have first been introduced by Friedman and Savage [10] and, apart from Hanson and Menezes [12], no many studies on risk attitude have referred to it until recently.

[2] See, for instance, Levy [15] on page 25.

The reason why future investment opportunities might be preferable is that investments are irreversible, as shown by the literature on investment under uncertainty (Dixit and Pindyck [5]). By delaying an investment, the investor can acquire additional information, which will then be used to decide how much to invest. In such situations, prudence cannot be related to the cost faced by the individual to accept the future outcome over the current one. In this paper, we show that prudence is actually related to the benefit drawn from the investment delay, which we will call a "flexibility gain", as in Dixit and Pindyck [5]. Intuitively, prudence is related to the flexibility gain because a delay in investment leads to a reduction in downside risk. By not engaging in an irreversible investment today, the investor preserves the possibility of investing less tomorrow, if an unfavorable state of nature is realized.

As an illustration, take an individual who obtains a net utility of $u(y) - \theta y$ from an immediate investment in a capacity of size y, where θ is the expected cost of acquiring a capacity unit and the true cost is $\theta + \widetilde{\eta}$, where $\widetilde{\eta} \in \{-\eta, \eta\}$ with equal probabilities. Taking the utility loss from investment to be linear looks as a simplification as compared to what a utility function should be, but it is helpful for illustrating what we would expect to observe with a more general utility function as well. The optimal choice of y is endogenous and depends on the production technology (as represented by θ and $\theta + \widetilde{\eta}$). Moreover, whereas we refer to y as to "capacity," y may well represent the amount of production that a principal asks an agent to produce, in a contract, when the cost of production is unknown; or it may represent the supply of a monopolist who knows the market demand, whereas the cost is unknown. Not surprisingly, assuming no discount factor, the decision maker will prefer to delay the decision until after she will have observed the realization of $\theta + \widetilde{\eta}$, hence she has a flexibility gain. We find that a greater flexibility gain for the individual is associated with a lower value of θ if and only if u''' is positive. This is explained by the preference for less downside risk, as in Menezes et al. [16], with the caveat that here less downside risk reflects a greater flexibility gain rather than a lower utility premium.

In the development, we consider a general utility function which depends on both the profit obtained by using the capacity and on the cost of acquisition, where the cost is subject to a stochastic shift between periods, as in the above example. As one might expect, the flexibility gain will depend on the characteristics of the cost function. However, the link between flexibility gain and prudence follows the same principle as in the previous example.

Looking next at the preference over different distributions of optimal capacities, we consider cost functions of the form $\theta_i + \widetilde{\eta}_j$, where ij denotes a distribution characterized by mean θ_i and spread $\widetilde{\eta}_j$. We show that the higher that u''' is, the more that distributions with lower θ_i and/or more dispersed values $\widetilde{\eta}_j$ are preferable to other distributions. Hence, the shape of the marginal utility of capacity provides a measure of how much the individual is available to pay to be able to use a better technology, such as a technology with a lower expected cost of acquisition. Remarkably, the distributions among which the individual

is required to choose respect neither first nor second order stochastic dominance, as most commonly assumed by the literature.

We also consider an optimal prevention problem, in line with Eeckhoudt and Gollier [6]. In a setting in which the individual must decide today how much to invest in order to prevent a disaster, the authors show that a prudent individual invests less than an imprudent individual would do. We extend their setting by allowing for the possibility of a stochastic change of the cost of effort from one period to another and before the disaster might occur. The individual has to choose between making the effort immediately and waiting for the next period, when the cost of effort will have evolved. We show that, in this kind of setting, a prudent individual has no flexibility gain. This is because, when the investment is to be made to prevent a disaster, rather than representing an "opportunity", delaying the investment adds more to the risk that the individual faces.

To complete the analysis, we provide applications to principal-agent models and investment timing problems. In principal-agent models in which the type of the agent is a distribution rather than a state of nature, the information rents depend on the third derivative of the utility function. In investment timing problems in which the individual may have a flexibility gain from delaying investments (i.e., investments represent an "opportunity" rather than a preventive effort), a prudent investor decides to invest less often.

The paper is first related to the studies belonging to the literature on Decision theory, in which prudence is defined as an averse attitude to downside risk (Menezes et al. [16], Bigelow and Menezes [1] and Eeckhoudt and Schlesinger [8]). We contribute to this literature by showing that, in addition to being associated with the cost borne by the individual when the future outcome is taken over the current one, the aversion to downside risk is also associated with akind of preference for investment delay.

The paper is also related to the studies in Decision theory that identify a link between prudence and irreversibility of decision making (Gollier et al. [11] and Eeckhoudt and Gollier [6]). However, this literature focuses exclusively on preventive investment problems and, particularly, on situations in which no information acquisition can take place before deciding to invest. We contribute to this literature by showing that prudence is also related to situations in which information can be acquired prior to investing.

In the literature on investment under uncertainty, the investor is usually assumed to be neutral to risk. This is without loss of generality as long as financial markets are complete and a risk averse investor can always find costless ways to hedge against the risk. An exception is the study of Henderson and Hobson [13], who allow for incomplete markets. Not surprisingly, they find that a risk averse investor invests less often under uncertainty than does a risk neutral investor. Hitherto no study has shown how an investment timing decision is affected by the fact that the individual is prudent. To make the point, in the main development we consider a setting in which uncertainty is resolved after one period, as is usual in Decision theory. In the application to an investment timing decision under

multi-period uncertainty, we rely on a specific example, in which the stochastic variable follows a random walk law of motion.

The outline of the paper is as follows. In Section 2 we present the model. In Section 3 we analyze the link between prudence and flexibility gain to the individual in a two-period model. In Section 4 we consider related concepts that are used in Decision theory and compare with the notion of flexibility gain. In Section 5 we present **the** applications. Section 6 concludes.

2   Model

Consider an individual who has an investment opportunity. The investment has a cost of $C(y,x)$ and yields a profit of $\pi(y)$, where y is the size of the activity (say, the installed capacity) and x is a variable that induces a stochastic shift of the cost function between periods. The overall utility of the individual is given by

$$\widehat{u}_1(m + \pi(y)) + \widehat{u}_0(m - C(y,x)),$$

where m is her initial wealth, $\widehat{u}_1(\cdot)$ is the utility obtained once the outcome of investment is obtained and $\widehat{u}_0(\cdot)$ is the utility of the individual at the moment when the cost of investment must be faced. The cost function is such that $\frac{\partial C(y,x)}{\partial y} > 0$, $\frac{\partial C(y,x)}{\partial x} > 0$ and $\frac{\partial^2 C(y,x)}{\partial y \partial x} > 0$. The latter two conditions express the role of the variable x in the model. For any given capacity y, the total cost and the marginal cost both increase with x. For later use, we also define the marginal rate of the cost $MRC(y,x)$ as the ratio between the marginal change of the cost with respect to x and the marginal change of the cost with respect to capacity y: $MRC_{x/y} = \frac{\partial C(y,x)}{\partial x} / \frac{\partial C(y,x)}{\partial y}$.

We assume that there are two periods and that the variable x is evolving intertemporally. We take $x \in \{\theta, \theta + \widetilde{\eta}\}$. If the investment is made in period zero, then the cost function is known and given by $C(y,\theta)$, with $\theta > 0$. If the investment is made in period 1, then the cost function is unknown in period zero and given by $C(y, \theta + \widetilde{\eta})$. We assume that $E[\widetilde{\eta}] = 0$, $\widetilde{\eta} \in [-\overline{\eta}, \overline{\eta}]$, where $\overline{\eta} > 0$, and that the distribution function $g(\widetilde{\eta})$ has positive density everywhere.

Because the utility functions ultimately depend on y, the size of the activity, we further introduce the notation $\widehat{u}_1(m + \pi(y)) = u_{1,m}(y)$ and $\widehat{u}_0(m - C(y,x)) = u_{0,m}(y,x)$, for convenience. The function $u_{1,m}(y)$ is such that $u'_{1,m}(y) > 0$, $u''_{1,m}(y) < 0$ and $u'''_{1,m}(y)$ has a constant sign. Similarly, $\widehat{u}_0(\cdot)$ is such that $\widehat{u}'_0(\cdot) > 0$, $\widehat{u}''_0(\cdot) < 0$ and $\widehat{u}'''_0(\cdot)$ has a constant sign. We respectively denote as $A(y) = -\frac{u''_{1,m}(y)}{u'_{1,m}(y)}$ and $P(y) = -\frac{u'''_{1,m}(y)}{u''_{1,m}(y)}$ the coefficient of absolute risk aversion and that of absolute prudence of the individual with respect to the capacity y.

Noticeably, here above, after stating the properties of $u_{1,m}(y)$, we have presented the

properties of $\widehat{u}_0(\cdot)$ rather than those of $u_{0,m}(y,x)$. Referring to $\widehat{u}_0(\cdot)$ is useful in that is enables us to distinguish between the utility of money and the cost function, which includes the stochastic component x, in addition to y.

3   Prudence and flexibility gain

There are two decisions that the individual has to make. First, the individual chooses whether to invest in period zero or wait for the realization of $\widetilde{\eta}$ before making that decision. Second, given $x \in \{\theta, \theta + \widetilde{\eta}\}$, the individual decides how many capacity units to install.

Let us first consider the second decision. The first-order condition of the individual's program is given by

$$u'_{1,m}(y) = -\frac{\partial u_{0,m}(y,x)}{\partial y},$$

where $u'_{1,m}(y)$ is the marginal benefit of y in period 1 and $-\frac{\partial u_{0,m}(y,x)}{\partial y} > 0$ is the marginal utility loss that the individual must face in period zero to acquire y units. We denote the unique solution as $y_m(x)$. Observing that $y'_m(x) = -MRC_{x/y}$, one has $y'_m(x) < 0$. Because a higher value of x reflects a higher cost of acquiring a given amount of capacity, less capacity will be installed. Assuming no discount factor, for simplicity, it is optimal to delay the investment decision until the next period, when $\widetilde{\eta}$ will be known, provided that $w(\theta) > 0$, where

$$w(\theta) = \mathbb{E}\left[u_{1,m}(y(\theta + \widetilde{\eta})) + u_{0,m}(y(\theta + \widetilde{\eta}), \theta + \widetilde{\eta})\right] - \left[u_{1,m}(y(\theta)) + u_{0,m}(y(\theta), \theta)\right].$$

We can say that, by delaying the decision to invest, the investor obtains a flexibility gain, which is given by the possibility of setting the capacity level according to the true realization of the cost. The existence of a flexibility gain is explained by the irreversible nature of the investment of cost $C(y,x)$. By committing today to acquire $y(\theta)$, the individual renounces to the opportunity of purchasing $y(\theta + \widetilde{\eta})$ instead of $y(\theta)$ capacity units, when $\widetilde{\eta}$ will be known and, implicitly, so will be the cost function. Provided $u_{0,m}(y,x) = \widehat{u}_0(m - C(y,x))$, the flexibility gain is found to be related to a quota of the utility $u_{0,m}(y,x)$ to which the individual must renounce to acquire capacity units. The lemma below shows that the flexibility gain is positive: the lower the value of x is, the more units of capacity the individual will want to acquire, hence the higher the marginal utility loss $-\frac{\partial u_{0,m}(y(x),x)}{\partial x}$ that she is ready to be faced with.

Lemma 1 *The flexibility gain is expressed as follows:*

$$w(\theta) = \int_{\theta-\eta}^{\theta}\left(-\frac{\partial u_{0,m}(y(x),x)}{\partial x}\right)g(x)\,dx - \int_{\theta}^{\theta+\eta}\left(-\frac{\partial u_{0,m}(y(x),x)}{\partial x}\right)g(x)\,dx \quad (1)$$

Proof. Considering that the optimal capacity is such that $u'_1(y(x)) = \partial u_0(y(x),x)/\partial y(x)$,

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$\forall x$, rewrite

$$
\begin{aligned}
w\left(\theta\right) & = \mathbb{E}\left[u_{1,m}\left(y\left(\theta+\widetilde{\eta}\right)\right)+u_{0,m}\left(y\left(\theta+\widetilde{\eta}\right),\theta+\widetilde{\eta}\right)\right]-\left[u_{1,m}\left(y\left(\theta\right)\right)+u_{0,m}\left(y\left(\theta\right),\theta\right)\right] \\
& = \mathbb{E}\left[\int_{\theta}^{\theta+\widetilde{\eta}}\left[\left(u_{1,m}'\left(y\left(x\right)\right)+\frac{\partial u_{0,m}\left(y\left(x\right),x\right)}{\partial y\left(x\right)}\right)y'\left(x\right)+\frac{\partial u_{0,m}\left(y\left(x\right),x\right)}{\partial x}\right]dx\right] \\
& = \mathbb{E}\left[\int_{\theta}^{\theta+\widetilde{\eta}}\frac{\partial u_{0,m}\left(y\left(x\right),x\right)}{\partial x}dx\right],
\end{aligned}
$$

which is further reformulated as (1). ∎

Resting on the above lemma, we will show that the marginal flexibility gain $\frac{dw(\theta)}{d\theta}$ depends on the absolute degree of prudence of the individual. To that end, we define $v_m\left(x\right)=u_{1,m}\left(y\left(x\right)\right)$, for convenience. Using the first-order condition, we see that the marginal loss $-\frac{\partial u_0(y(x),x)}{\partial x}$ incurred by the individual when investing is tantamount to a marginal utility of $-v_m'\left(x\right)$, which is obtained as the installed capacity is used. Indeed,

$$
\begin{aligned}
-v_m'\left(x\right) & = -u_{1,m}'\left(y\left(x\right)\right)y'\left(x\right)=u_{1,m}'\left(y\left(x\right)\right)MRC_{x/y(x)} \\
& = \widehat{u}_0'\left(m-C\left(y\left(x\right),x\right)\right)\frac{\partial C\left(y\left(x\right),x\right)}{\partial x} \\
& = -\frac{\partial u_0\left(y\left(x\right),x\right)}{\partial x}.
\end{aligned}
$$

Recall now that the flexibility gain exists because the individual is available to face a decreasing marginal loss of $-\frac{\partial u_0(y(x),x)}{\partial x}$. This translates into a decreasing marginal benefit of $-v'\left(x\right)$ from using the installed capacity. Take now the marginal flexibility gain $\frac{dw(\theta,\overline{\eta})}{d\theta}$. A raise in θ induces a downward shift of the marginal loss $-\frac{\partial u_0(y(x),x)}{\partial x}$, $\forall x\in\left[\theta-\overline{\eta},\theta+\overline{\eta}\right]$. This reflects a downward shift of the marginal benefit from investment, $-v'\left(x\right)$, $\forall x\in\left[\theta-\overline{\eta},\theta+\overline{\eta}\right]$. Moreover, the decrease in the marginal benefit is not necessarily equal to the left and to the right of θ. Actually, it depends on $-\left[v_m'\left(x\right)g\left(x\right)\right]''$, i.e., on the shape of $-v'\left(x\right)$ as weighted by the density function of each specific value of x. Knowing that the flexibility gain is expressed as in (1), that shape determines whether the flexibility gain decreases or increases with θ.

Lemma 2 *Assume that $\left[v'\left(x\right)g\left(x\right)\right]''$ has constant sign. Then, $\frac{dw(\theta,\overline{\eta})}{d\theta}<0$ if and only if*

$$
-\left[v_m'\left(x\right)g\left(x\right)\right]''<0.
$$

Proof. Using (1), we calculate

$$
\begin{aligned}
\frac{dw\left(\theta,\overline{\eta}\right)}{d\theta} & = \left[\frac{\partial u_{0,m}\left(y\left(\theta+\overline{\eta}\right),\theta+\overline{\eta}\right)}{\partial\theta}g\left(\theta+\overline{\eta}\right)-\frac{\partial u_{0,m}\left(y\left(\theta\right),\theta\right)}{\partial\theta}g\left(\theta\right)\right] \\
& \quad -\left[\frac{\partial u_{0,m}\left(y\left(\theta\right),\theta\right)}{\partial\theta}g\left(\theta\right)-\frac{\partial u_{0,m}\left(y\left(\theta-\overline{\eta}\right),\theta-\overline{\eta}\right)}{\partial\theta}g\left(\theta-\overline{\eta}\right)\right].
\end{aligned}
$$

Using $u_{0,m}(y(x),x) = \widehat{u}_0(m - C(y(x),x))$, this is further rewritten as

$$
\begin{aligned}
\frac{dw(\theta,\overline{\eta})}{d\theta} =\ & \widehat{u}_0'(m - C(y(\theta+\overline{\eta}),\theta+\overline{\eta})) \frac{\partial C(y(\theta+\overline{\eta}),\theta+\overline{\eta})}{\partial\theta} g(\theta+\overline{\eta}) \\
& - \widehat{u}_0'(m - C(y(\theta),\theta)) \frac{\partial C(y(\theta),\theta)}{\partial\theta} g(\theta) \\
& - \widehat{u}_0'(m - C(y(\theta),\theta)) \frac{\partial C(y(\theta),\theta)}{\partial\theta} g(\theta) \\
& + \widehat{u}_0'(m - C(y(\theta-\overline{\eta}),\theta-\overline{\eta})) \frac{\partial C(y(\theta-\overline{\eta}),\theta-\overline{\eta})}{\partial\theta} g(\theta-\overline{\eta}).
\end{aligned}
$$

Reformulating the first-order condition as

$$
\begin{aligned}
u_{1,m}'(y(x)) =\ & \widehat{u}_0'(m - C(y(x),x)) \frac{\partial C(y(x),x)}{\partial y(x)} \\
\Leftrightarrow\ & \widehat{u}_0'(m - C(y(x),x)) = \frac{u_{1,m}'(y(x))}{\frac{\partial C(y(x),x)}{\partial y(x)}},
\end{aligned}
$$

we can write

$$
\begin{aligned}
\frac{dw(\theta,\overline{\eta})}{d\theta} =\ & u_{1,m}'(y(\theta+\overline{\eta})) \frac{\frac{\partial C(y(\theta+\overline{\eta}),\theta+\overline{\eta})}{\partial\theta}}{\frac{\partial C(y(\theta+\overline{\eta}),\theta+\overline{\eta})}{\partial y(\theta+\overline{\eta})}} g(\theta+\overline{\eta}) - u_{1,m}'(y(\theta)) \frac{\frac{\partial C(y(\theta),\theta)}{\partial\theta}}{\frac{\partial C(y(\theta),\theta)}{\partial y(\theta)}} g(\theta) \\
& + u_{1,m}'(y(\theta-\overline{\eta})) \frac{\frac{\partial C(y(\theta-\overline{\eta}),\theta-\overline{\eta})}{\partial\theta}}{\frac{\partial C(y(\theta-\overline{\eta}),\theta-\overline{\eta})}{\partial y(\theta-\overline{\eta})}} g(\theta-\overline{\eta}) - u_{1,m}'(y(\theta)) \frac{\frac{\partial C(y(\theta),\theta)}{\partial\theta}}{\frac{\partial C(y(\theta),\theta)}{\partial y(\theta)}} g(\theta) \\
=\ & \int_{\theta-\overline{\eta}}^{\theta} \int_{x}^{x+\overline{\eta}} \left[u_{1,m}'(y(z)) MRC(y(z),z) g(z) \right]'' dzdx,
\end{aligned}
$$

where the definition $MRC(y,z) = \frac{\partial C(y,z)}{\partial z} / \frac{\partial C(y,z)}{\partial y}$ is used. Observing from this same definition that $y'(z) = -MRC(y,z)$, we can further write

$$
\frac{dw(\theta,\overline{\eta})}{d\theta} = \int_{\theta-\overline{\eta}}^{\theta} \int_{x}^{x+\overline{\eta}} \left[u_{1,m}'(y(z)) y'(z) g(z) \right]'' dxdz = \int_{\theta-\overline{\eta}}^{\theta} \int_{x}^{x+\overline{\eta}} \left[v_m'(z) g(z) \right]'' dxdz.
$$

The result in the lemma follows because $\left[v_m'(z) g(z) \right]''$ has constant sign. ∎

From the above lemma, it is clear that the flexibility gain is related to the third derivative of the utility function. We are left with showing in which way. According to the next result, the marginal flexibility gain is negative if and only if the coefficient of absolute prudence is sufficiently high, depending on the properties of the cost function.

Proposition 1 $\frac{dw(\theta,\overline{\eta})}{d\theta} < 0$ *if and only if:*

$$
P_1(y(z)) > \frac{1}{A_1(y(z))} \left(-\frac{[MRS(z) g(z)]''}{MRS(z) g(z)} \right) + 2 \frac{[MRS(z) g(z)]'}{MRS(z) g(z)}
$$

Proof. Compute

$$v'_m(x) = u'_{1,m}(y(x)) y'(x) = -u'_{1,m}(y(x)) \frac{\frac{\partial C(y(x),x)}{\partial x}}{\frac{\partial C(y(x),x)}{\partial y(x)}}.$$

Then

$$
\begin{aligned}
[v'_m(z) g(z)]'' &= -\left[u'_{1,m}(y(z)) MRS(z) g(z)\right]'' \\
&= -u'''_{1,m}(y(z)) MRS(z) g(z) - 2u''_{1,m}(y(z)) [MRS(z) g(z)]' \\
&\quad -u'_{1,m}(y(z)) (MRS(z) g(z))''.
\end{aligned}
$$

Because $MRS(z) > 0$, $[v'_m(z) g(z)]'' < 0$ if and only if

$$-\frac{u'''_{1,m}(y(z))}{u''_{1,m}(y(z))} > 2\frac{[MRS(z) g(z)]'}{MRS(z) g(z)} + \frac{u'_{1,m}(y(z))}{u''_{1,m}(y(z))} \frac{[MRS(z) g(z)]''}{MRS(z) g(z)},$$

which is rewritten as in the proposition. ∎

Therefore, the benefit of the decision to delay the investment is related to how prudent the individual is. In investment problems, the benefit of information acquisition is usually more important the more costly that the investment is. What Proposition 1 shows is that this is true when the individual is not very prudent, but not otherwise. To understand the importance of this result for individual decision making, we compare it with the ways in which the concept of prudence has been used in the literature and develop a few economic applications.

4 Related concepts

4.1 Utility premium

In this section we show that the result in Proposition 1 allows for a broader interpretation of prudence than usually adopted in the literature. We know that, by applying Kimball's definition, prudence is equivalent to $u'''(\cdot) > 0$, where $u(\cdot)$ is the utility function of some individual. Applying Jensen inequality, this is equivalent to $E[u'(y + \widetilde{\varepsilon})] > u'(y)$, for some random $\widetilde{\varepsilon}$ such that $E[\widetilde{\varepsilon}] = 0$. Using this equivalence, prudence is usually associated with the utility premium the individual requires for accepting the unknown outcome. Indeed, supposing that the future capacity is $y + \widetilde{\varepsilon}$ and that it is *exogenously given* to the individual at no cost, she is prudent when $w'_0(y) > 0$, where

$$w_0(y) = \mathbb{E}[u(y + \widetilde{\varepsilon})] - u(y) < 0,$$

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and $-w_0(y)$ is her utility premium (see, for instance, Eeckhoudt and Schlesinger [8]). For a prudent individual the utility premium $-w_0(y)$ decreases as y is raised. If the individual has to pay for that capacity and her utility is quasi-linear in the cost occasioned by that purchase, then the utility premium is $-w_1(y)$, where

$$w_1(y) = \mathbb{E}[u(y + \widetilde{\varepsilon})] - u(y) - \{\mathbb{E}[(\theta + \widetilde{\varepsilon})(y + \widetilde{\varepsilon})] - \theta y\}.$$

Because $\mathbb{E}[(\theta + \widetilde{\varepsilon})(y + \widetilde{\varepsilon})] - \theta y = \mathbb{E}[\widetilde{\varepsilon}^2]$, the utility premium is negative and even lower than in the previous case, if $\widetilde{\varepsilon}$ has a strictly positive variance.

Let us now turn to consider our setting, where y is *endogenous*. To compare with the approach just presented, we simplify the model in the previous section by assuming that $g(\cdot)$ is a constant, which entails that the distribution function is uniform, that the utility $u_0(\cdot)$ is quasilinear, $\widehat{u}_0(m - xy) = m - xy$, and that the cost function is linear: $C(y, x) = xy$. As we will see, this simplification will allow us to provide a measure of the marginal flexibility gain as depending on the sign of $u'''(\cdot)$. The individual has a flexibility gain from delaying the investment, written as:

$$
\begin{aligned}
\widehat{w}(\theta) &= \mathbb{E}[u_{1,m}(y(\theta + \widetilde{\eta})) - (\theta + \widetilde{\eta})y(\theta + \widetilde{\eta})] - (u_{1,m}(y(\theta)) - \theta y(\theta)) \\
&= \mathbb{E}\int_{\theta}^{\theta + \widetilde{\eta}} [u'_{1,m}(y(x)) - xy'(x) - y(x)]\, dx.
\end{aligned}
$$

Knowing that the optimal capacity $y(x)$ is such that $u'_{1,m}(y(x)) = xy'(x)$, $\forall x$, we can rewrite the above expression as

$$\widehat{w}(\theta) = \int_{\theta - \eta}^{\theta} [y(x) - y(x + \eta)]\, \overline{g}\, dx, \tag{2}$$

where $\overline{g} = g(x)$, $\forall x$. Unlike $w_0(y)$ and $w_1(y)$, $\widehat{w}(\theta)$ is positive. Rather than facing a cost from taking the outcome under the expected operator, instead of the certain one, the individual obtains a benefit, which represents the flexibility gain. Recall from the above presentation that the sign of the marginal utility premium depends on whether the individual is or not prudent. We show in what follows that also the marginal flexibility gain depends on whether the individual is or not prudent.

Corollary 1 $\widehat{w}'(\theta) < 0$ *if and only if* $u'''_{1,m}(\cdot) > 0$.

Proof. Define $f(\cdot)$ the inverse function of $u'_{1,m}(\cdot)$ and

$$\zeta(a, b, c) = [f(a) - f(a + c)] - [f(b) - f(b + c)], \tag{3}$$

where $b > a$ and $c > 0$. One has $\zeta(a, b, c) > 0$ if and only if $\int_a^{a+c} [f'(z) - f'(z + b - a)]\, dz < 0$. Provided $u'''_{1,m}(\cdot)$ has constant sign, this is also the case of $f'(\cdot)$, and $\zeta(a, b, c) > 0$ if and only if $f'(z) < f'(z + b - a)$, for any given $z \in [a, a + c]$. This is equivalent to

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$u_{1,m}''\left(y\left(z\right)\right) > u_{1,m}''\left(y\left(z + b - a\right)\right)$. Because $b > a$ and $u_{1,m}''\left(\cdot\right) < 0$, $y\left(z\right) > y\left(z + b - a\right)$. Hence, $u_{1,m}''\left(y\left(z\right)\right) > u_{1,m}''\left(y\left(z + b - a\right)\right)$ is equivalent to $u''' > 0$ and so is $\zeta\left(a, b, c\right) > 0$.

Using (2), we can write

$$
\begin{aligned}
\widehat{w}'\left(\theta\right) &= \left\{\left[y\left(\theta\right) - y\left(\theta + \eta\right)\right] - \left[y\left(\theta - \eta\right) - y\left(\theta\right)\right]\right\}\overline{g} \\
&= -\zeta\left(\theta - \eta, \theta, \eta\right)\overline{g}.
\end{aligned}
$$

Hence, $\widehat{w}'\left(\theta\right) < 0$ if and only if $\zeta\left(\theta - \eta, \theta, \eta\right) > 0$, which was shown to be equivalent to $u''' > 0$. ∎

In good substance, the reason why the flexibility gain is related to whether the individual is or not prudent is that the downside risk is reduced when the investment (and, implicitly, the consumption) is delayed rather than taking place immediately. This provides a different approach to prudence than usually thought of.

4.2   Choice between distributions

As is well known, the preference of the individual derived with the expected utility approach can be replicated as a choice between distributions rather than a choice between a certain outcome today and an uncertain outcome tomorrow. For that, the distributions must respect some stochastic ordering. Indeed, an individual whose utility $u\left(y\right)$ increases with y dislikes facing the distribution $F_2\left(y\right)$ which is first-order stochastically dominated (FSD) by $F_1\left(y\right)$, since low values of y are more likely in $F_2\left(y\right)$ than in $F_1\left(y\right)$, whereas the converse is true for high values. If the individual is also risk averse, then she dislikes facing a distribution $F_2\left(y\right)$ that is second-order stochastically dominated (SSD) by $F_1\left(y\right)$, provided $F_2\left(y\right)$ is riskier than $F_1\left(y\right)$. Moreover, a prudent individual dislikes $F_2\left(y\right)$ if it is third-order stochastically dominated (TSD) by $F_1\left(y\right)$, provided $F_2\left(y\right)$ embodies a higher downside risk than $F_1\left(y\right)$. The literature has shown that this reasoning applies to the comparison between higher order derivatives of the utility function and higher orders of stochastic dominance (see Eeckhoudt and Schlesinger [8]).

Consider now the notion of prudence and its link with stochastic dominance. Considering that FSD implies SSD and that SSD implies TSD, it is sufficient to learn that $F_2\left(y\right)$ is FSD by $F_1\left(y\right)$, or SSD by $F_1\left(y\right)$, to know that a prudent individual will prefer the lottery $F_1\left(y\right)$ to $F_2\left(y\right)$. However, when this is true, one does not learn anything specific about prudence, since risk aversion is sufficient for the individual to prefer the distribution $F_1\left(y\right)$. In our approach, in which the individual is an investor and enjoys a flexibility gain in her investment decision, the preference over distributions is related to the notion of prudence for distributions that respect neither first-order nor second-order stochastic dominance. This is shown hereafter.

Take for simplicity the net utility function $u_{1,m}\left(y\right) - xy$, as in the previous section, and assume that $x \in \left\{\theta_i + \overline{\eta}_j,\ \theta_i - \overline{\eta}_j\right\}$, where the two values have equal probability, $\forall i \in \left\{1, 2\right\}$ and $\forall j \in \left\{1, 2\right\}$, $\theta_2 > \theta_1$ and $\overline{\eta}_2 > \overline{\eta}_1$. Instead of choosing whether to invest in period

zero or one, the individual must select one of four cost distributions, that are indexed by ij. Remarkably, two distributions have the same mean and two of them the same spread, so one can associate the former pair with first-order stochastic dominance and the latter pair with mean-preserving spread, which is a particular case of second-order stochastic dominance. However, neither first- nor second-order stochastic dominance applies, if the four distributions are considered altogether.

Turning now to the preference of the individual over these distributions, she will obviously prefer $1j$ to $2j$, and it is easy to show that she also prefers $i2$ to $i1$. We will prove that the third derivative of the utility function measures the extent to which the individual prefers to be faced with any of these distributions. To that end, we define

$$D_{ij/i'j'} = \mathbb{E}\left[U_{ij} - U_{i'j'}\right],$$

where

$$U_{ij} = u_{1,m}\left(y\left(\theta_i + \widetilde{\eta}_j\right)\right) - \left(\theta_i + \widetilde{\eta}_j\right) y\left(\theta_i + \widetilde{\eta}_j\right).$$

For instance, $D_{1j/2j}$ measures the additional gain that a technology associated with an expected unit cost of θ_1 grants, relative to one associated with an expected unit cost of θ_2, for any given value η_j of the spread.

Proposition 2 (i) $D_{1j/2j'}$ increases with $u_{1,m}'''\left(\cdot\right)$, $\forall j \geq j'$.
(ii) $D_{i2/i'1}$ increases with $u_{1,m}\left(\cdot\right)'''$, $\forall i \leq i'$.
(iii) $D_{11/22}$ increases with $u_{1,m}\left(\cdot\right)'''$ if and only if $\theta_2 - \theta_1 > \eta_2 - \eta_1$.

Proof. Being based on the definition of U_{ij}, we can compute

$$
\begin{aligned}
D_{ij/i'j'} &= \mathbb{E}\left[U_{ij} - U_{i'j'}\right] \\
&= -\mathbb{E}\left[\int_{\theta_i+\widetilde{\eta}_j}^{\theta_{i'}+\widetilde{\eta}_{j'}} u'\left(y\left(x\right)\right) y'\left(x\right) dx\right] - \mathbb{E}\left[\left(\theta_i + \widetilde{\eta}_j\right) y\left(\theta_i + \widetilde{\eta}_j\right)\right] \\
&\quad + \mathbb{E}\left[\left(\theta_{i'} + \widetilde{\eta}_{j'}\right) y\left(\theta_{i'} + \widetilde{\eta}_{j'}\right)\right] \\
&= -\mathbb{E}\int_{\theta_i+\widetilde{\eta}_j}^{\theta_{i'}+\widetilde{\eta}_{j'}} xy'\left(x\right) dx - \mathbb{E}\left[\left(\theta_i + \widetilde{\eta}_j\right) y\left(\theta_i + \widetilde{\eta}_j\right)\right] + \mathbb{E}\left[\left(\theta_{i'} + \widetilde{\eta}_{j'}\right) y\left(\theta_{i'} + \widetilde{\eta}_{j'}\right)\right],
\end{aligned}
$$

which further reduces to

$$D_{ij/i'j'} = \mathbb{E}\int_{\theta_i+\widetilde{\eta}_j}^{\theta_{i'}+\widetilde{\eta}_{j'}} y\left(x\right) dx. \tag{4}$$

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Proof of (i) and (ii). Using (4) and (3), we can further develop

$$
\begin{aligned}
D_{1j/2j} &= \mathbb{E} \int_{\theta_1+\widetilde{\eta}_j}^{\theta_2+\widetilde{\eta}_j} y(x)\, dx \\
&= \mathbb{E}\left[\int_{\theta_1+\widetilde{\eta}_j}^{\theta_1} y(x)\, dx + \int_{\theta_1}^{\theta_2} y(x)\, dx + \int_{\theta_2}^{\theta_2+\widetilde{\eta}_j} y(x)\, dx \right] \\
&= \frac{1}{2}\int_{\theta_1-\eta_j}^{\theta_1} \zeta(x, x+\Delta\theta, n_j)\, dx + \int_{\theta_1}^{\theta_2} y(x)\, dx,
\end{aligned}
$$

where $\Delta\theta = \theta_2 - \theta_1$. In the expression of $D_{1j/2j}$, the term that depends on $u_{1,m}'''(\cdot)$ is $\zeta(\cdot,\cdot,\cdot)$, **as** defined in (3), **which** increases with $u_{1,m}'''(\cdot)$. Also,

$$
\begin{aligned}
D_{i2/i1} &= -\mathbb{E}\left[\int_{\theta_i+\widetilde{\eta}_1}^{\theta_i+\widetilde{\eta}_2} y(x)\, dx \right] \\
&= \frac{1}{2}\left[\int_{\theta_i-\eta_2}^{\theta_i} y(x)\, dx - \int_{\theta_i}^{\theta_i+\eta_2} y(x)\, dx \right] - \frac{1}{2}\left[\int_{\theta_i-\eta_1}^{\theta_i} y(x)\, dx - \int_{\theta_i}^{\theta_i+\eta_1} y(x)\, dx \right] \\
&= \frac{1}{2}\int_{\theta_i-\eta_2}^{\theta_i} \zeta(x, x+\Delta\eta, \eta_2)\, dx + \frac{1}{2}\left[\int_{\theta_i}^{\theta_i+\Delta\eta} y(x)\, dx - \int_{\theta_i+\eta_1}^{\theta_i+\eta_1+\Delta\eta} y(x)\, dx \right],
\end{aligned}
$$

where $\Delta\eta = \eta_2 - \eta_1$. Again, the term that depends on $u_{1,m}(\cdot)'''$ is $\zeta(\cdot,\cdot,\cdot)$, which increases with $u_{1,m}(\cdot)'''$.

Rewriting

$$
D_{1j/2j'} = D_{1j/2j} + D_{2j/2j'} = \begin{cases} D_{1j/2j}, & \text{if } j' = j, \\ D_{12/22} + D_{22/21}, & \text{if } j' = 1 < j = 2 \end{cases}
$$

and

$$
D_{i2/i'1} = D_{i2/i1} + D_{i1/i'1} = \begin{cases} D_{i2/i1}, & \text{if } i = i', \\ D_{12/11} + D_{11/21}, & \text{if } i = 1 < i' = 2 \end{cases},
$$

and considering that $D_{1j/2j}$, $\forall j$, and $D_{i2/i1}$, $\forall i$, increase with $u_{1,m}(\cdot)'''$, we deduce that this is the case of $D_{1j/2j'}$ and $D_{i2/i'1}$ as well.

Proof of (iii). Using (4) and (3), we can compute

$$
\begin{aligned}
D_{11/22} &= \mathbb{E}\left[\int_{\theta_L+\widetilde{\eta}_L}^{\theta_H+\widetilde{\eta}_H} y(x)\, dx \right] \\
&= \frac{1}{2}\int_{\theta_L-\eta_L}^{\theta_H-\eta_H} \zeta(x, x+\eta_L, x+2\eta_L)\, dx + \int_{\theta_L}^{\theta_L+\Delta\theta-\Delta\eta} y(x)\, dx
\end{aligned}
$$

If $\Delta\theta > \Delta\eta$, then $D_{11/22}$ is positive and increases with $\zeta(\cdot,\cdot,\cdot)$; if $\Delta\theta < \Delta\eta$, then it is negative and decreases with $\zeta(\cdot,\cdot,\cdot)$. Provided that $\zeta(\cdot,\cdot,\cdot)$ increases with $u_{1,m}'''(\cdot)$, the result follows. ∎

Therefore, how much an individual prefers being faced with a set $\{\theta_i - \eta_j, \theta_i + \eta_j\}$,

rather than with a set $\left\{ \theta_{i'} - \eta_{j'}, \theta_{i'} + \eta_{j'} \right\}$, is related to $u'''_{1,m}(\cdot)$. Remarkably, one cannot rely on the usual notions of stochastic dominance to capture the link between a preference over distributions and $u'''_{1,m}(\cdot)$. Indeed, it is easy to show that, if two distributions share the same support but are respectively ordered in the sense of first-order stochastic dominance and mean preserving spread, then the preference of the individual for one distribution is unrelated to $u'''_{1,m}(\cdot)$.

4.3 Optimal prevention

As explained by Gollier and Eeckhoudt [6], prevention is defined as the effort undertaken to reduce the probability of occurrence of an adverse effect. They find that a prudent individual makes a lower preventive effort than an imprudent individual. In their setting, the individual has no current consumption and the probability of future consumption depends on the preventive effort. The preventive effort is "inflexible" in the sense that there is no benefit from information acquisition by postponing the decision. In what follows we show that when the investment has the nature of a preventive effort, a prudent individual does not consider the information acquisition as an opportunity and, hence, there is no flexibility gain.

We begin by considering the model of Gollier and Eeckhoudt [6]. The individual may or may not be exposed to a loss of $L > 0$ in the future. The probability of being exposed to the loss, denoted $p(e)$, depends on her effort e. The expected utility is written as

$$p(e) u(m - L - e) + [1 - p(e)] u(m - e).$$

Letting $c(e)$ be the cost of exerting effort, we can say that, in this formulation, $c(e) = e$.

Let us assume now that the individual may take advantage of the information that becomes available over time before deciding how much effort to exert. This is possible because the cost of effort is $c_0(e) = e$ at time zero and $c_1(e) = e + \widetilde{\eta}$ at time one. The flexibility gain from delaying the decision is written as

$$
\begin{aligned}
w_p = \ & \mathbb{E}\left[p(e_1(\widetilde{\eta})) u(m - L - e_1(\widetilde{\eta}) - \widetilde{\eta}) + (1 - p(e_1(\widetilde{\eta}))) u(m - e_1(\widetilde{\eta}) - \widetilde{\eta}) \right] \\
& - \left[p(e_0) u(m - L - e_0) + (1 - p(e_0)) u(m - e_0) \right],
\end{aligned}
$$

where e_0 is the optimal effort, if exerted in period zero, and $e_1(\widetilde{\eta})$ is the optimal effort, if exerted in period one, which depends on the realization of $\widetilde{\eta}$.

Proposition 3 *If* $u''(\cdot) = 0$ *then* $w_p = 0$.

If $u''(\cdot) < 0$ *and* $u'''(\cdot) \geq 0$, *then* $w_p < 0$.

Proof. The optimal effort is such that

$$u\left(m-L-x\right)-u\left(m-x\right)=\frac{p\left(e\right)}{p'\left(e\right)}u'\left(m-L-x\right)+\frac{1-p\left(e\right)}{p'\left(e\right)}u'\left(m-x\right),$$

where $x=e$ if the effort is made in period zero, and $x=e+\widetilde{\eta}$ if effort is made in period one. Using this first-order condition in w_p, we can rewrite

$$
\begin{aligned}
w_p &= \mathbb{E}p\left(e\right)\left[\frac{p\left(e\right)}{p'\left(e\right)}u'\left(m-L-e-\widetilde{\eta}\right)+\frac{1-p\left(e\right)}{p'\left(e\right)}u'\left(m-e-\widetilde{\eta}\right)\right] \\
&\quad -p\left(e\right)\left[\frac{p\left(e\right)}{p'\left(e\right)}u'\left(m-L-e\right)+\frac{1-p\left(e\right)}{p'\left(e\right)}u'\left(m-e\right)\right] \\
&\quad +\mathbb{E}\left[u\left(m-e-\widetilde{\eta}\right)\right]-u\left(m-e\right).
\end{aligned}
$$

This is further reformulated as

$$
\begin{aligned}
w_p &= \frac{p\left(e\right)}{p'\left(e\right)}\int_{e-\overline{\eta}}^{e}\int_{x}^{e+\overline{\eta}}p\left(e\right)\frac{e+\overline{\eta}-x}{\overline{\eta}}u'''\left(m-L-z\right)+\left(1-p\left(e\right)\right)\frac{e+\overline{\eta}-x}{\overline{\eta}}u'''\left(m-z\right)dzdx \\
&\quad +\int_{e-\overline{\eta}}^{e}\int_{x}^{e+\overline{\eta}}u''\left(m-z\right)dx,
\end{aligned}
$$

from which the results in the proposition follow. ∎

The proposition is explained by the fact that, when the investment to be made consists in a preventive effort, delaying the investment raises the risk to which the investor is exposed rather than granting him an opportunity. This is why a risk neutral individual attaches no value to the possibility of acquiring information, and for a risk averse individual with non-concave marginal utility the flexibility gain is negative. This result shows that the core reason why prudence is related to the flexibility gain in our model, is that future brings better investment opportunities rather than more risk.

5 Applications

We hereafter propose a few examples, in which the utility function of the decision maker is given by $u\left(y\right)-xy$, where $u\left(y\right)$ is a constant relative risk aversion function, defined as follows:

$$u\left(y\right)=\frac{1}{1-\gamma}y^{1-\gamma},$$

for some $\gamma\in\left(0,1\right)$. Accordingly, we have $u'=y^{-\gamma}>0$, $u''=-\gamma y^{-\gamma-1}<0$ and $u'''=\gamma\left(\gamma+1\right)y^{-\gamma-2}>0$. Considering a utility function with these properties is convenient in that it permits to look at variations in u''' through variations in γ. Indeed, one has

$$\frac{du'''}{d\gamma}=\left(2\gamma+1\right)y^{-\gamma-2}+\gamma\left(\gamma+1\right)y^{-\gamma-2}\ln y,$$

which is strictly positive if the quantity y is above one. The second and the third derivatives of $u(\cdot)$ are also the second and the third derivatives of $u(y) - xy$.

5.1 Delegation with unknown cost in the contracting stage

A principal who delegates the production activity to an agent obtains the gross utility $u(y)$ from consumption of the y units produced by the agent. By assigning a profit of $\pi = t(x, y) - xy$ to the agent, where $t(x, y)$ is a transfer and x the unit cost of production, the principal obtains a net utility of $u(y) - xy - \pi$.

Take $x_0 = \theta$ to be known and $x_1 \in \{\theta - \eta, \theta + \eta\}$ with equal probabilities. From the previous analysis, we know that, if $\pi = 0$, then the principal strictly prefers to condition the production quantity on x_1, rather than on x_0, since, by doing so, she obtains a greater flexibility gain. Moreover, according to Corollary 1, higher values of θ are associated with a smaller flexibility gain because $u''' > 0$.

Remarkably, in principal-agent models the usual concept that is used in order to express the principal's preferences is that of the monotonic likelihood ratio, which is more restrictive than stochastic dominance (see, for instance, Eeckhoudt et al [7], page 39). We hereafter show that, in the framework just considered, it is essential to refer to the concept of prudence.

Suppose that the principal can choose between an agent producing at a cost of $\theta_1 + \widetilde{\eta}_1$ and an agent producing at a cost of $\theta_2 + \widetilde{\eta}_2$, where $\theta_1 < \theta_2$ and $\eta_1 < \eta_2$. We saw that, if the principal can leave zero profit to the agent regardless of his cost, then the principal prefers a cost of $\theta_1 + \widetilde{\eta}_1$ if and only if $\theta_2 - \theta_1 > \eta_2 - \eta_1$ (Proposition 2). Moreover, the gain increases with u'''. Indeed, using $u' = \theta + \widetilde{\eta}$ and $u' = y^{-\gamma}$, we see that $y(\theta + \widetilde{\eta}) = (\theta + \widetilde{\eta})^{-\frac{1}{\gamma}}$ and

$$D_{11/22} = \mathbb{E} \int_{\theta_1 + \widetilde{\eta}_1}^{\theta_2 + \widetilde{\eta}_2} x^{-\frac{1}{\gamma}} dx$$

together with

$$\frac{dD_{11/22}}{d\gamma} = \frac{1}{\gamma^2} \mathbb{E} \int_{\theta_1 + \widetilde{\eta}_1}^{\theta_2 + \widetilde{\eta}_2} x^{-\frac{1}{\gamma}} \ln(x) \, dx,$$

which is positive if the unit cost is above 1 in all states. Since $du'''/d\gamma > 0$, we can say that a greater value of u''' is associated with a greater value of $D_{11/22}$, as in Proposition 2.

Why is this relevant? Suppose that the principal does not know the type of agent she is facing, where the type is defined by the distribution indexed by $ij \in \{11, 12, 21, 22\}$. Instead, the agent holds this information. Applying the Revelation Principle, the agent is required to make a report to the principal about his type, and a rent must be conceded to type 11 in order not to declare, for instance, 22. The greater u''' is the higher the information rent that the principal will prefer to concede to type $\theta_1 + \widetilde{\eta}_1$ to solve the usual trade-off between rent extraction and efficiency loss. Indeed, denoting Π_1 and Π_2 the profits designed for the

two types and setting $\Pi_2 = 0$, it is easy to verify that

$$\Pi_1 = (\theta_2 - \theta_1)\,\mathbb{E}\left[y\left(\theta_2 + \widetilde{\eta}_2\right)\right] - (\eta_2 - \eta_1)\,\mathbb{E}\left[y\left(\theta_2 - \eta_2\right) - y\left(\theta_2 + \eta_2\right)\right].$$

Replacing $y\left(\theta + \widetilde{\eta}\right) = \left(\theta + \widetilde{\eta}\right)^{-\frac{1}{\gamma}}$, this becomes

$$\Pi_1 = (\theta_2 - \theta_1)\,\mathbb{E}\left[\left(\theta_2 + \widetilde{\eta}_2\right)^{-\frac{1}{\gamma}}\right] - (\eta_2 - \eta_1)\,\mathbb{E}\left[\left(\theta_2 - \eta_2\right)^{-\frac{1}{\gamma}} - y\left(\theta_2 + \eta_2\right)^{-\frac{1}{\gamma}}\right].$$

Therefore, we see that

$$
\begin{aligned}
\frac{d\Pi_1}{d\gamma} &= \frac{1}{2\gamma^2}\left((\theta_2 - \theta_1) - (\eta_2 - \eta_1)\right)\left(\left(\theta_2 - \eta_2\right)^{-\frac{1}{\gamma}}\ln\left(\theta_2 - \eta_2\right)\right) \\
&\quad + \frac{1}{2\gamma^2}\left((\theta_2 - \theta_1) + (\eta_2 - \eta_1)\right)\left(y\left(\theta_2 + \eta_2\right)^{-\frac{1}{\gamma}}\ln\left(\theta_2 + \eta_2\right)\right) \\
&> 0
\end{aligned}
$$

which confirms that the principal prefers to assign a higher information rent the greater u''' is.[34]

5.2 Price discrimination with unknown preferences

The example presented above can be framed within the recent literature on principal-agent problems with privately known distributions. Most of those studies are about price discrimination in the relationship between a monopolist and a consumer, none of whom knows the consumer's valuation for the good in the contracting stage, whereas the consumer has private information on the distribution of his valuation. The pioneering study is that of Courty and Li [2]. They assume that the monopolist receives a fixed payment a when the consumer is still uninformed of his valuation. This might be followed by a reimbursement k, which the consumer can require in a later stage, after learning his true valuation. Of course, the consumer will want to be reimbursed, and will thus renounce to consume, if and only if k exceeds his valuation. If the consumer does not renounce, then the monopolist will bear a cost of c to provide the service.

---

[3]Although little apparent from our presentation, there is also an additional aspect of the incentive problem which is related to u'''. That is, if both θ and η are privately known to the agent, then a greater u''' will also reflect the fact that adjacent incentive constraints are tighter than other incentive constraints. For instance, reporting $\theta_1 + \widetilde{\eta}_1$ is more attractive to a type $\theta_1 + \widetilde{\eta}_2$ than reporting $\theta_2 + \widetilde{\eta}_2$. A complete analysis is developed by Danau and Vinella [3], who show that the study of the optimal delegation in this context is lengthy and complicated, unless u''' is considered.

[4]Remarkably, the results in the delegation example here proposed extend naturally to a problem of monopoly regulation, if $u\left(\cdot\right)$ is interpreted as consumer surplus. Accordingly, in that framework, $u'\left(\cdot\right)$ would measure the consumer willingness to pay for the good sold by the monopolist and, hence, the (inverse) demand for the good. Variations in u''' would represent variations in the price elasticity of the market demand rather than variations in the preferences of a risk averse decision maker. One would find that $\gamma = 1/\varepsilon_p$, where ε_p is the constant price elasticity of the demand. (A: is this still ok after we arranged slightly computations in the Sequential paper? perhaps better check) Therefore, a greater value of u''' would be associated with a less elastic demand and the results we presented in the example follow accordingly.

Essentially, in Courty and Li [2], the economic issue is how to choose the future disbursement k and the current revenue a, which is more in line with the classical savings-consumption model than with the issue of our interest. However, because this problem belongs to the kind of principal-agent models considered in the previous example, for the sake of completeness, we show that u''' plays a role in the solution adopted by the principal also in this case. To that end, we restrict attention to the case of symmetric information between players.

Whereas Courty and Li [2] and more recent studies assume that the monopolist is risk neutral, we consider a risk averse monopolist, whose utility $u(\cdot)$ is expressed as a function of money, as defined above. The total benefit of the monopolist is

$$u(a) - vu(k) - (1 - \nu) u(c),$$

where ν is the probability of a high valuation, namely $v > 0$, $(1 - \nu)$ is the probability of a low valuation, namely 0, and the reimbursement k is supposed to take values in $(0, v)$ at optimum. Under symmetric information, if the risk neutral consumer has zero outside opportunity, then the monopolist chooses a fixed payment such that

$$a(k) = (1 - \nu) k + \nu v.$$

The first-order condition of the maximization problem of the principal is given by

$$u'(k + \nu(v - k)) \geq \frac{\nu}{1 - \nu} u'(k).$$

For a positive solution to exist, it is necessary and sufficient that the high valuation is less likely than the low valuation: $v < 1/2$. Otherwise, the monopolist will choose $a = \nu v$ without conceding any reimbursement. Accordingly, we take $\nu < 1/2$. Then, $k > 0$ involving that $a(k) > \nu v$. Replacing $u'(y) = y^{-\gamma}$, we obtain the following solution:

$$[k + \nu(v - k)]^{-\gamma} = \frac{v}{1 - v} k^{-\gamma} \Leftrightarrow k^* = \frac{\nu v}{\left(\frac{1 - v}{v}\right)^{\frac{1}{\gamma}} - (1 - \nu)},$$

which is lower than v and confirms our previous hypothesis. We see that $dk^*/d\gamma > 0$. Hence, the greater u''' is the higher the value k^* that the solution takes. This is interpreted as follows. A more prudent monopolist is more prone to grant a reimbursement to the consumer in a later stage to be able to appropriate a higher certain payment $a(k^*)$ today.

5.3 Investment timing and prudence

We now come back to the simple capacity investment model presented in section 3, assuming that the value of x evolves stochastically as a simple random walk. We take

$x_{t+1} \in \{x_t - \eta, x_t + \eta\}$, $\forall t$, where $t \in \mathbb{N}_+$ indexes the period. We also assume that there is a discount factor between each two periods, namely $\delta \in (0, 1)$. The flexibility gain from increasing the delay of investment from t to $t + 1$ is written as

$$w_t (x_t) = \delta \mathbb{E} [u (y (x_t + \widetilde{\eta})) - (x_t + \widetilde{\eta}) y (x_t + \widetilde{\eta})] - (u (y (x_t)) - x_t y (x_t)).$$

Assuming that, at a generic time t, the investment has not been made yet, the investor will delay the investment if and only if $w_t (x_t) > 0$. In other words, the investment is made at the first time t at which x_t is such that $w_t (x_t) \leq 0$. The bigger $w_t (x_t)$ is the more the investment is expected to be delayed over time. We will check the relationship between $w_t (x_t)$ and γ to assess the role that u''' plays in the investment timing decision.

Using the first-order condition $u' (y (x)) = x$ and $u' (y) = y^{-\gamma}$ we obtain $y (x)^{-\gamma} = x \Leftrightarrow y (x) = x^{-\frac{1}{\gamma}}$. Hence,

$$u (y (x)) = u \left(x^{-\frac{1}{\gamma}} \right) = \frac{x^{-\frac{1-\gamma}{\gamma}}}{1 - \gamma}.$$

Therefore,

$$w_t (x_t) = \frac{\delta \gamma}{1 - \gamma} E \left[(\theta + \widetilde{\eta})^{-\frac{1}{\gamma}+1} \right] - \frac{\gamma \theta^{-\frac{1-\gamma}{\gamma}}}{1 - \gamma}$$

and $w_t (x_t) > 0$ if and only if

$$\mathbb{E} \left[(\theta + \widetilde{\eta})^{-\frac{1-\gamma}{\gamma}} \right] > \frac{\theta^{-\frac{1-\gamma}{\gamma}}}{\delta} \Leftrightarrow \left(\frac{\theta}{\theta + \eta} \right)^{\frac{1-\gamma}{\gamma}} + \left(\frac{\theta}{\theta - \eta} \right)^{\frac{1-\gamma}{\gamma}} > \frac{2}{\delta}$$

Computing

$$\frac{d}{d\gamma} \left[\left(\frac{\theta}{\theta + \eta} \right)^{\frac{1-\gamma}{\gamma}} + \left(\frac{\theta}{\theta - \eta} \right)^{\frac{1-\gamma}{\gamma}} \right] = -\frac{1}{\gamma^2} \left[\ln \left(\frac{\theta}{\theta + \eta} \right)^{\left(\frac{\theta}{\theta + \eta} \right)^{\frac{1-\gamma}{\gamma}}} \left(\frac{\theta}{\theta - \eta} \right)^{\left(\frac{\theta}{\theta - \eta} \right)^{\frac{1-\gamma}{\gamma}}} \right],$$

we see that it is positive. Hence, the more convex the third derivative of the utility function is, the more likely it is that the investment decision will be delayed.[5]

6 Conclusion

We showed that the notion of prudence extends to situations the literature has not considered so far. Specifically, provided that an individual prefers future outcomes to current ones, the third derivative of the utility function (and, implicitly, the degree of prudence) is a

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[5]Noticeably, if the investment has the nature of a preventive effort, then, using the same utility function as in the applications we consider, one should find that, with $u''' (\cdot) > 0$, the flexibility gain is negative, as in Proposition 3. We did not develop this case because the optimal effort cannot be expressed explicitly, given that the first-order condition is a complex polinomial function. However, as from Proposition 3, prudence is a sufficient condition for the flexibility gain to be negative in that case, regardless of the utility function specifically considered.

measure of that preference. This is explained by the fact that when information acquisition arises over time, downside risk is lower **as** decisions are delayed.

The comparisons with the literature we developed and the applications we proposed show why it is useful to consider the broader interpretation of prudence we identified. In principal-agent models, it may be necessary to know the third derivative of the surplus function of the principal in order to identify which incentive constraints are tighter and may be binding. In investment timing problems, prudence induces investment delay. However, when the nature of the investment is that of a preventive effort, a prudent individual has a strict preference for immediate investment. These applications helped us illustrate that the notion of prudence can be employed along a novel research direction.

References

[1] Bigelow, J. P., and Carmen F. Menezes (1995), "Outside Risk Aversion and the Comparative Statics of Increasing Risk in Quasi-Linear Decision Models", International Economic Review, 36(3), 643-73

[2] Courty, P., and H. Li (2000), "Sequential Screening," Review of Economic Studies, 67, 697-717

[3] Danau, D., and A. Vinella (2016), "Sequential screening and the relationship between principal's preferences and agent's incentives", SERIES Working Papers, Dipartimento di Scienze economiche e metodi matematici, Università degli Studi di Bari Aldo Moro, N. 01/2016

[4] Dixit, A. (1993), "The art of smooth pasting", in Fundamentals of Pure and Applied Economics 55, Harwood Academic Publishers

[5] Dixit, A., and R. S. Pindyck (1994), "Investment under uncertainty", Princeton University Press

[6] Eeckhoudt, L., and C Gollier (2005), "The impact of prudence on optimal prevention", *Economic Theory,* 26, 989–994

[7] Eeckhoudt, L., C. Gollier and H. Schlesinger (2005), "Economic and financial decisions under risk", Princeton University Press

[8] Eeckhoudt, L., and H. Schlesinger (2006), "Putting Risk in Its Proper Place", *American Economic Review*, 96 (1), 280-289

[9] Eeckhoudt, L., and H. Schlesinger (2006), "On the utility premium of Friedman and Savage", Economics Letters, 105 (1), 46-48

[10] Friedman, M. and L.J. Savage (1948), "The Utility Analysis of Choices Involving Risk.", *Journal of Political Economy*, 56(4), 279-304

[11] Gollier, C., B. Jullien and N. Treich (2000), "Scientific progress and irreversibility: an economic interpretation of the 'Precautionary Principle' ", *Journal of Public Economics,* 75, 229–253

[12] Hanson, L., and C. F. Menezes (1971), "On a Neglected Aspect of the Theory of Risk Aversion", Western Economic Journal, 9, 211-17

[13] Henderson, V., and D. G. Hobson (2002), "Real options with constant relative risk aversion", Journal of Economic Dynamics & Control, 27, 329–355

[14] Leland, H. (1968), "Saving and Uncertainty: The Precautionary Demand for Saving", The Quarterly Journal of Economics, 2, 465-473

[15] Levy, H. (2016), "Stochastic Dominance. Investment Decision Making under Uncertainty, Third edition, Springer

[16] Menezes,C., C. Geiss and J. Tressler (1980), "Increasing Downside Risk", The American Economic Review, Vol. 70, 5, 921-932

[17] Sandmo, A. (1970), "The Effect of Uncertainty on Saving Decisions", The Review of Economic Studies, 37: 353-360

[18] Kimball, M. (1990), "Precautionary Saving in the Small and in the Large", Econometrica, 58 (1): 53-73