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On the optimal use of correlated information in contractual design under limited liability

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Abstract

Riordan and Sappington (JET, 1988) show that in an agency relationship in which the type of the agent is correlated with a signal that is observed publicly *ex post*, the principal may attain first best (full surplus extraction and efficient output levels) if she offers the agent a lottery such that each type is rewarded for one signal realization and punished equally for all the others. Gary-Bobo and Spiegel (RAND, 2006) show that this kind of lottery is most likely to be locally incentive-compatible when the agent is protected by limited liability. In this paper we investigate how the principal should construct the lottery to attain not only local but also global incentive-compatibility. We first assess that the main issue with global incentive-compatibility rests with intermediate types being potentially attractive reports to both lower- and higher-order types. We then show that a lottery including three (rather than two) levels of profit is most likely to be globally incentive-compatible under limited liability, if local incentive constraints are strictly satisfied. We identify conditions under which first best is implemented and pin down the optimal distortions when those conditions are violated. In particular, when the first-best allocation is locally but not globally incentive-compatible, output distortions are induced but no information rent is conceded to the agent.

Keywords: Incentive compatibility; Limited liability; Correlated signals; Conditional probability; Full-rank condition

J.E.L. Classification Numbers: D82

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1 Introduction

There is now notable work on contractual design in agency problems with correlated information. The pioneering studies, which we owe to Myerson [9], Crémer and McLean [2] (henceforth, CM), McAfee and Reny [8] and Riordan and Sappington [10] (henceforth, RS), identify sufficient conditions for full surplus extraction in settings in which the agent is not protected by limited liability. When such conditions are satisfied, the principal designs a payment scheme including a lottery related to the distribution of an external signal to be realized *ex post* and correlated with the private information of the agent. All surplus is extracted from the agent by embedding in the lottery both rewards and punishments associated with the various possible signal realizations. However, a serious drawback of these mechanisms is that the punishments may be too high for the mechanisms to be viable when the agent is protected by limited liability.

Demougin and Garvie [3] and Gary-Bobo and Spiegel [5] (henceforth, GBS) investigate optimal screening under limited liability in the presence of correlated information. Demougin and Garvie [3] only consider the case in which the signal is binary. GBS show that this is without loss of generality when the principal is only concerned with local incentive-compatibility, in addition to limited liability. In that case, indeed, the principal is better off if she offers a lottery that admits only two levels of profit, a reward and a punishment. If more than two signals are available, then the reward is associated with only one signal and the punishment with all the others. However, it is not obvious that this is still the best strategy in environments in which global incentive-compatibility is not implied by local incentive-compatibility. Hitherto the literature has not completely clarified which exact lottery the principal should adopt when incentive-compatibility may be difficult to attain not only locally but also globally and the agent is protected by limited liability. Here is the contribution of our study.

RS show that a lottery yielding a reward for a single signal realization is optimally adopted when the cost function of the agent is less concave in type than the conditional likelihood function of the reward signal. GBS focus on situations in which the cost function is strictly convex in type whereas the conditional likelihood function of the reward signal is concave in type, which are sufficient conditions for first-best implementation in light of the results of RS. In this study, restrictions are imposed neither on the curvature of the cost function nor on that of the conditional likelihood function of the signal to which the highest profit is associated. On the one hand, in so doing, we allow for the total cost function to be concave in type, as may well be the case, for instance, if the agent has an affine cost function such that the fixed cost is inversely related to the privately known marginal cost.[1] On the other hand, we admit that there exist two signals (rather than only one, as in GBS) which, taken together with any of the other available signals, satisfy the monotonic likelihood property. Although we reinforce the assumption made by GBS in this respect, we nonetheless require the monotonic

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[1]This is highly plausible in regulated sectors, in which low marginal costs are often associated with high overhead costs (Maggi and Rodriguez-Clare [7]). Alternatively, one could think of better outside opportunities as being associated with more efficient production (Lewis and Sappington [6]).

likelihood property, which is familiar in mechanism design, to hold only in a "partial" sense, not necessarily in a complete sense. With this approach we can search for the best lottery that the principal could employ to implement first best under limited liability. Our results will depend on how the shape of the cost function compares with that of the conditional likelihood function, as in RS, but the family of cost functions for which full surplus extraction is at hand is likely to be richer than in the one-reward lottery scheme. More specific results are summarized hereafter.[2]

Overview of the results

We first show that the main difficulty with global incentive-compatibility is rooted in the way in which the lotteries targeted to the intermediate types should be designed for those types to represent attractive reports neither to lower- nor to higher-order types. This is better understood if it is considered that the compensation to the agent blends together a fixed payment related to the cost of production and a lottery related to the *ex-post* realization of the signal. On the one hand, lower-order types exaggerating information gain on the fixed payment but lose in terms of lottery; on the other, higher-order types under-stating information gain in terms of lottery but lose on the fixed payment. This double circumstance constrains the principal when designing the lotteries for the intermediate types.[3]

Second, when local incentive-compatibility is attained under limited liability, the lottery that is most likely to be globally incentive-compatible at the first-best allocation includes three distinct levels of profit for each type of agent. One profit is a reward associated with the signal displaying the highest likelihood ratio; a second profit is a punishment associated with all other signals but that displaying the lowest likelihood ratio; and the third profit is an intermediate profit associated with this latter signal. This articulated structure of the lottery is feasible because the principal enjoys some flexibility when the liability of the agent is sufficiently high to be able to attain local incentive-compatibility with a lottery other than the one of GBS. Specifically, the losses inflicted to the intermediate types can be differentiated across signal realizations in such a way that over-statement by lower-order types is more easily prevented, yet, without making under-statement significantly more attractive to higher-order types. This facilitates the task of the principal to impede that intermediate types be conveniently announced by any other type.

Third, once the lottery which is most likely to attain global incentive-compatibility under limited liability is characterized, a cut-off level of liability is determined, which dictates whether or not first best is implementable. This cut-off value depends on how the shape of the cost function with respect to type compares with the shape of the likelihood function of the reward

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[2]In GBS the exogenous signal is taken to affect the cost of production, rather than being purely informational about that cost, as is usually assumed by the literature. We do not follow the approach of GBS to avoid introducing complications that are unnecessary to the purpose of our study.

[3]From the proofs of Corollary 1.4 and 1.5 of RS it emerges that first best is implementable once sufficient conditions are introduced under which there is no conflict between incentive constraints. However, no study has thereafter clarified why such a conflict may arise and how it can be eliminated under limited liability.

signal. Put it differently, the exact family of cost functions for which first best is viable is determined, given the level of liability. For instance, first best is at reach if the cost function of the agent is concave in type, rather than being convex as in GBS, but the degree of concavity is not too pronounced relative to that of the likelihood function of the reward signal.

Our fourth finding concerns situations in which the liability of the agent is too low for the principal to induce truthtelling without distortions, and hence for the first-best allocation to be effected. We show that the structure of the optimal lottery in this second-best scenario does not differ from that figured out in the first-best setting. An important aspect is that the level of liability which separates the regime under which local incentive-compatibility is attained from that under which it is not, is also the level of liability which separates situations in which the optimal lottery includes three levels of profit from those in which, as in GBS, it includes only two levels of profit. Remarkably, in the former situations, inducing distortions in the volume of output is the only instrument the principal can use to satisfy both upward and downward incentive constraints, which are then binding. No type is assigned any information rent.

Related literature

First of all, our paper is related to Myerson [9], CM and McAfee and Reny [8], who consider an environment in which a seller/principal auctions out an object to a number of potential buyers/agents whose preferences (types) are privately known and correlated. In that environment, the signals correlated with the type of each agent are generated endogenously by the reports collected by the principal from the other agents. From those studies we know that the principal retains all surplus in a Bayesian framework, for any *unspecified utility function* of each agent, if and only if the vector of conditional probabilities of the type of any agent, given the types of the other agents, is linearly independent of the vector of conditional probabilities of the types of the other agents. Whereas this result is very appealing in contractual design, it nonetheless exhibits the aforementioned limit of inducing compensations that are potentially very low. It may thus be difficult to attain in practice.

A second line of research to which the paper is related is pioneered by RS. They consider situations in which the principal deals with only one agent whose private information is correlated with some signal which is realized and publicly observed *ex post*. These are thus situations in which the correlated signals are exogenous to the contractual relationship. However, provided that the external signals play the same role as the private information held by other agents, RS obtain a result similar to that derived by the first line of research. In addition, RS show that, for some *specific cost functions* of the agent, full surplus extraction is at hand in spite of the informational signals being less numerous than the possible types of the agent.[4] More precisely, whether or not the outcome is attainable depends on the relationship between the characteristics of the cost function of the agent and the properties of the likelihood functions

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[4]In CM the types of the agents (the potential buyers of the object sold by the principal) determine their utilities. In RS, as in our study, the agent exerts an activity delegated by the principal and his type determines his cost of production.

of the signals. The most "parsimonious" lottery that the principal can design in this context includes only two levels of profit. GBS show that the incentive scheme proposed by RS is most likely to satisfy the limited liability constraints because the punishments are spread equally among all signals but one. With our investigation we evidence that this is not necessarily the best lottery the principal can use because there are circumstances under which it fails to motivate some types of agent to release information correctly. We then highlight how the lottery should be amended to circumvent these difficulty.

Our work is also related to the study of Demougin and Garvie [3], who were the first to analyze contractual design in situations in which correlated information becomes available *ex post* and the agent is protected by limited liability. In their model, this may mean, first, that the agent cannot be exposed to any loss, which is tantamount to imposing *ex-post* participation constraints. Alternatively, the transfers from the principal to the agent cannot be negative. In line with GBS, our approach is a generalization of the former kind of limited liability. We depart from the analysis of Demougin and Garvie [3] by allowing for more than two informational signals being available, which is essential for the results we draw.

As is well known, limited liability can alternatively be regarded as an extreme form of risk aversion. With that interpretation, our study is also related to the literature on full surplus extraction in agency problems with correlated information and risk aversion on the agent's side. Within this literature, Eso [4] considers an auction in which the auctioneer/principal faces two potential buyers/agents, both risk averse. Their privately known valuations of the object offered for sale are correlated and can take only two values. By contrast, we develop the analysis considering a richer set of types. This extension enables us to capture the important circumstance that incentive-compatibility is problematic essentially because intermediate types may potentially attract false reports from both below and above.

1.1 Outline

The reminder of the article is organized as follows. Section 2 describes the model. Section 3 presents the first-best analysis. We first look at a discrete number of types and then allow for a continuum of types. In Section 4, we investigate the second-best setting in which the level of liability is too low, or the cost function too concave in type, to implement the first-best allocation. Section 5 concludes. Mathematical proofs are relegated to an appendix.

2 The model

A principal, P, makes a contractual offer to an agent for the production of a good (or service). They are both risk neutral. Consumption of q units of the good yields a gross utility of $S(q)$. The function $S(\cdot)$ is twice continuously differentiable and such that $S'(\cdot) > 0$, $S''(\cdot) < 0$, $S(0) = 0$ and the Inada's conditions are satisfied. The cost of producing q units amounts to $C(q, \theta)$, where the "type" θ parametrizes the productivity of the agent. A

lower value of θ involves a lower total cost for any given q and will be referred to as a lower-order (more efficient) type. The function $C(\cdot,\cdot)$ is twice continuously differentiable in either argument with derivatives $dC(q,\theta)/dq \equiv C_q(q,\theta) > 0$, $(dC(q,\theta)/d\theta) \equiv C_\theta(q,\theta) > 0$ and $(d^2C(q,\theta)/dqd\theta) = C_{q\theta}(q,\theta) > 0$, the latter meaning that less efficient types have higher marginal costs of production. The agent receives a payment of t for the supply of q units of the good.

In the contracting stage, the agent knows his type whereas P does not. It is commonly known that θ is drawn from the support $\Theta \equiv [\underline{\theta},\overline{\theta}]$, where $\overline{\theta} > \underline{\theta} > 0$ with continuously differentiable density function $f(\theta)$ and cumulative distribution function $F(\theta)$. Alternatively, θ is known to take values in the discrete set $\Theta_T \equiv \{\theta_1,...\theta_T\}$, where T is the number of types. This alternative scenario will sometimes be considered for analytical and expositional purposes; it will also be useful to present previous findings of the literature and develop comparisons. Notation will be adapted accordingly whenever necessary.

The type of the agent is correlated with a random signal s, which is realized and publicly observed *ex post*, *i.e.*, after the contract is drawn up and the level of output is determined (or the output is produced). The signal is "hard" information involving that a legally enforceable contract can be signed upon.[5] We take the signal to be drawn from the discrete support $N \equiv \{1,..,n\}$, where $n \geq 2$. The probability that signal s is realized conditional on the agent's type being θ is $p_s(\theta)$. We assume that $p_s(\theta) > 0 \ \forall s \in N$ and that the function $p_s(\cdot)$ is twice continuously differentiable for all values of θ, with first and second derivative respectively denoted $(dp_s(\theta)/d\theta) \equiv p'_s(\theta)$ and $(d^2p_s(\theta)/d\theta^2) \equiv p''_s(\theta)$. We also assume that, for any triplet of signals $\{1,s,n\}$, the following monotonicity property is satisfied:

$$\frac{p_1(\theta)}{p_1(\theta')} > \frac{p_s(\theta)}{p_s(\theta')} > \frac{p_n(\theta)}{p_n(\theta')}, \ \forall \theta > \theta', \ \forall s \neq 1,n. \tag{1}$$

The Revelation Principle applies and P offers a menu of allocations $\{q(\theta),t_s(\theta)\}_{\forall \theta,\forall s}$, where $q(\theta)$ is the quantity an agent of type θ is required to produce and $t_s(\theta)$ is the transfer he is assigned when signal s is realized. The quantity is not conditioned on the signal because it is chosen (or the output is produced) prior to the signal realization. The net surplus of P is $S(q(\theta)) - t_s(\theta)$. Denote $\widetilde{\pi}_s(\theta'|\theta) \equiv t_s(\theta') - C(q(\theta'),\theta)$ the profit an agent of type θ obtains when he announces θ' to P. Also let $\pi_s(\theta) = \widetilde{\pi}_s(\theta|\theta)$ so that we can write:

$$\widetilde{\pi}_s(\theta'|\theta) = \pi_s(\theta') + C(q(\theta'),\theta') - C(q(\theta'),\theta). \tag{2}$$

Further denote $\boldsymbol{\pi}(\theta) \equiv \{\pi_s(\theta)\}_{\forall s}$ the lottery that the agent is faced with. We shall say that he receives a *reward* if $\pi_s(\theta) > 0$, he incurs a *punishment* if $\pi_s(\theta) < 0$. Before proceeding to the analysis, it is useful to remark that (2) would be the same if $t_s(\theta)$ were to include a fixed

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component related to the type and a stochastic component conditional on the signal realization, as considered by Bose and Zhao [1]. Consistent with this, the programme of P presented below only depends on the profits rather than on the exact structure of the transfers assigned to the various types when the different signals are realized.

2.1 The programme of the principal

Referring to the profit $\pi_s(\theta)$ rather than to the transfer $t_s(\theta)$ with a standard change of variable, the programme of P is formulated as follows:

$$\max_{\{q(\theta);\boldsymbol{\pi}_s(\theta)\}_{\forall\theta}} W \equiv \int_{\underline{\theta}}^{\overline{\theta}} \sum_{s=1}^{n} \left(S(q(\theta)) - C(q(\theta),\theta) - \pi_s(\theta)\right) p_s(\theta) \, dF(\theta)$$

subject to

$$\mathbb{E}_s\left[\pi_s(\theta)\right] \geq 0, \ \forall\theta \tag{PC}$$

$$\mathbb{E}_s\left[\pi_s(\theta)\right] \geq \sum_{s=1}^{n} \pi_s(\theta') p_s(\theta) + C(q(\theta'),\theta') - C(q(\theta'),\theta), \ \forall\theta,\theta' \tag{IC}$$

$$\pi_s(\theta) \geq -L, \ \forall\theta, \ \forall s. \tag{LL}$$

(PC) is the participation constraint whereby an agent of type θ incurs no loss in expectation. (IC) is the incentive-compatibility constraint whereby he is unwilling to report $\theta' \neq \theta$. (LL) is the limited liability constraint which ensures that the maximum loss to which he is exposed does not exceed $L > 0$ regardless of the signal realization.

3 First best

The first part of our study will be devoted to investigate under what conditions and in which way P implements the first-best allocation. This is defined by the optimality condition:

$$S'(q(\theta)) = C_q(q(\theta),\theta), \ \forall\theta, \tag{3}$$

together with the rent-extraction constraint:

$$\sum_{s=1}^{n} \pi_s(\theta) p_s(\theta) = 0, \ \forall\theta. \tag{4}$$

Throughout this section, to save on notation, $q(\theta)$ will indicate the first-best quantity for an agent of type θ and $\pi_s(\theta)$ the profit assigned for the production of that quantity. We further denote $\boldsymbol{\Pi}(\theta)$ the set of lotteries $\boldsymbol{\pi}(\theta)$, the elements of which satisfy (4).

3.1 Previous findings

Before turning to the analysis, it is useful to summarize the previous findings on first-best implementation in settings with correlated information.

RS Assume that θ_t takes values in the discrete set Θ_T, that $C(q, \theta_t)$ is convex in θ_t and that $\exists i \in N$ such that $p_i(\theta_t)$ is increasing and concave in θ_t. If $L \to \infty$, *i.e.* the agent can be exposed to unbounded losses, then $\boldsymbol{\Pi}(\theta_t)$ is not empty for any θ_t. After presenting this result in Corollary 1.4, RS show that P effects the first-best allocation by adopting the binary lottery $\boldsymbol{\pi}^i(\theta_t) \; \forall \theta_t$, defined as follows for any $t > 1$:

$$\pi_i(\theta_t) = [C(q(\theta_t), \theta_t) - C(q(\theta_t), \theta_{t-1})] \frac{1 - p_i(\theta_t)}{p_i(\theta_t) - p_i(\theta_{t-1})} \tag{5}$$

$$\pi_s(\theta_t) = -[C(q(\theta_t), \theta_t) - C(q(\theta_t), \theta_{t-1})] \frac{p_i(\theta_t)}{p_i(\theta_t) - p_i(\theta_{t-1})}, \quad \forall s \neq i. \tag{6}$$

In Corollary 1.5, RS further show that if $n = 2$, types are drawn from the discrete set Θ_3 and $p_i(\theta_3) > p_i(\theta_2) > p_i(\theta_1)$, then the lottery $\boldsymbol{\pi}^i(\theta_t)$ belongs to $\Pi(\theta_t)$ if and only if:

$$\frac{C(q, \theta_2) - C(q, \theta_1)}{C(q, \theta_3) - C(q, \theta_2)} \leq \frac{p_i(\theta_2) - p_i(\theta_1)}{p_i(\theta_3) - p_i(\theta_2)}. \tag{7}$$

This is ensured if the cost function is less concave in type than the conditional probability of signal i.

GBS Take $C(q, \theta)$ to be convex in θ and $p_i(\theta)$ to be increasing and concave in θ for some $i \in N$. Moreover, $i = \arg \max_{s \in N} \frac{p'_s(\theta)}{p_s(\theta)}, \; \forall \theta$. That is, among all possible signals and for all possible types, signal i is the one the probability of which displays the highest rate of change as type increases. Notice that under this assumption the condition that RS impose on signal i in their Corollary 1.5 is satisfied as well. Then, among all the lotteries belonging to $\boldsymbol{\Pi}(\theta)$, the one defined here below is the most likely to satisfy (LL):

$$\pi_i(\theta) = C_\theta(q(\theta), \theta) \frac{1 - p_i(\theta)}{p'_i(\theta)} \tag{8}$$

$$\pi_s(\theta) = C_\theta(q(\theta), \theta) \frac{p_i(\theta)}{-p'_i(\theta)}, \quad \forall s \neq i. \tag{9}$$

This is the counterpart of the lottery $\boldsymbol{\pi}^i(\theta_t)$ figured out by RS, for the case of a continuum of types.[6] Being based on (9), we see that the first-best allocation is implemented if and only if:

$$C_\theta(q(\theta), \theta) \frac{p_i(\theta)}{p'_i(\theta)} \leq L, \quad \forall \theta. \tag{10}$$

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[6] With a slight abuse of notation we will use the notation $\boldsymbol{\pi}^i(\cdot)$ to indicate this lottery regardless of whether types are drawn from a discrete set or a continuum range.

CM   Take $\theta_t \in \Theta_T$ and $L \to \infty$. As long as the vectors $\mathbf{p}(\theta_t)$ are linearly independent across types, $\mathbf{\Pi}(\theta_t)$ is non-empty for all θ_t. This follows from Farkas' lemma, which implies that there exists a $n-$dimensional vector $\mathbf{h}(\theta_t)$ $\forall \theta_t \in \Theta_T$, such that the following two conditions hold:

$$\sum_{s=1}^{n} h_s(\theta_t) p_s(\theta_t) = 0, \ \forall \theta_t \in \Theta_T \tag{11}$$

$$\sum_{s=1}^{n} h_s(\theta_t) p_s(\theta_{t'}) < 0, \ \forall \theta_t, \theta_{t'} \in \Theta_T. \tag{12}$$

By setting $\pi_s(\theta_t) = \gamma_t h_s(\theta_t), \ \forall s, \forall t,$ and choosing the "scaling" parameter γ_t arbitrarily big, P extracts all surplus from type θ_t and no incentive to mimic θ_t is triggered for any other type. First best is beyond reach if there exists some type θ_t for which a vector $\mathbf{h}(\theta_t)$ satisfying (11) and (12) does not exist.[7]

In good substance, RS highlight that, as long as the agent can be imposed unlimited punishments, first best is possibly at hand even when the set of informational signals includes only two elements. As is evident from the definition of $\boldsymbol{\pi}^i(\theta_t),$ the agent's gain only depends on whether signal i is realized, rather than any other signal, regardless of how rich the subset of other signals is. From GBS we further retain that any other lottery belonging to $\mathbf{\Pi}(\theta)$ includes an element the value of which is below that of (6), involving that it is less likely to satisfy (LL). Under assumption (1), $i = 1$ in our framework. The best known result in agency problems with correlated information is perhaps that of CM, who show that the first-best outcome is attained if the vectors of conditional probabilities of the signals are linearly independent. Importantly, this result is obtained regardless of the properties of the cost function. By setting rewards and punishments arbitrarily high, any report can be made unattractive to any other type. However, high punishments are unfeasible when the agent is protected by limited liability. One then needs to consider the properties of the cost and the probability functions to ascertain whether there exists some lottery that implements first best under limited liability, consistent with the analysis developed by GBS.

Our goal is to extend the analysis beyond that of GBS and investigate whether first best is attainable when (7) and (10) are not jointly satisfied and what lottery should be adopted in that case. Indeed, under assumption (1), (10) is most likely to hold for signal $i = 1$ but the associated lottery $\boldsymbol{\pi}^1(\theta)$ may fail to comply with (7) as required by RS. Whereas the assumption that some signal displays the highest likelihood ratio for all types is similar to that introduced by GBS, the assumption that some other signal displays the lowest likelihood ratio, also embodied in (1), is made for the purpose of our study. Overall, the conditions in (1) entails that the full-rank condition of CM must be satisfied for the extreme types but not necessarily

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[7] The "only if" proof of CM shows that if the vector $\mathbf{h}(\overline{\theta})$ does not exist, then it is impossible to prevent all types but $\overline{\theta}$ from mimicking type $\overline{\theta}$. Notice however that the full-rank condition is not necessary for all types. In particular, it does not need to hold for type $\underline{\theta}$. This paves the way to the results drawn in the study of RS, in which first-best implementation does not necessarily depend on the full-rank condition. Bose and Zhao [1] show that Proposition 1 in RS implies that first best might be effected when the full-rank condition is violated.

for the intermediate types.[8] In this respect, our analysis diverges from that of CM and comes closer to that of RS and GBS.

3.2 Three types and two or three signals

Before exploring the general setting with a continuum of types, it is useful to consider the case where types are drawn from the discrete set Θ_3 and are such that $\theta_1 < \theta_2 < \theta_3$. After looking at the basic scenario with a binary signal, we admit three signals in preparation to the subsequent analysis. Mathematical derivations are relegated to Appendix B.

3.2.1 Two signals

We take $N = \{1, 2\}$ and explore how the profits $\pi_1(\theta_t)$ and $\pi_2(\theta_t)$ of the generic type θ_t should be set. To that end, it is useful to proceed as follows. First construct a lottery that ensures full surplus extraction and identify the expression of the two profits included in that lottery. Then verify what changes the two profits should undergo when switching to a new lottery that preserves full surplus extraction.

Consider any lottery $\boldsymbol{\pi}(\theta_t) \in \Pi(\theta_t)$ under which $\pi_2(\theta_t) < 0 < \pi_1(\theta_t)$. Knowing that full surplus extraction requires the expected value of the lottery to be zero, we can write $\pi_1(\theta_t) = -\pi_2(\theta_t) \frac{p_2(\theta_t)}{p_1(\theta_t)}$ to express the lottery in terms of $\pi_2(\theta_t)$ only. The same can be done with the lottery that type $\theta_{t'}$ is faced with when reporting θ_t. Knowing that the likelihood of signal s conditional on the type being $\theta_{t'}$ is $p_s(\theta_{t'})$, one component of this lottery is $\pi_2(\theta_t) p_2(\theta_{t'})$, the other component is $-\pi_2(\theta_t) \frac{p_2(\theta_t)}{p_1(\theta_t)} p_1(\theta_{t'})$. Taking $\boldsymbol{\pi}(\theta_{t'})$ to belong to $\Pi(\theta_t)$ as well, type $\theta_{t'}$ faces a payoff of $\pi_2(\theta_t) p_2(\theta_t) \left(\frac{p_2(\theta_{t'})}{p_2(\theta_t)} - \frac{p_1(\theta_{t'})}{p_1(\theta_t)} \right)$ when reporting θ_t. This payoff is negative if $\theta_{t'} < \theta_t$; it is positive in the converse case. Therefore, a type below θ_t faces a lottery with lower expected value whereas a type above θ_t faces a lottery with higher expected value when cheating rather than telling the truth. This is explained by the fact that, under assumption (1), the rate of increase/decrease of $p_1(\cdot)$ as type increases/decreases is higher than that of $p_2(\cdot)$:

$$
\begin{aligned}
\frac{p_1(\theta_{t'}) - p_1(\theta_t)}{p_1(\theta_t)} &< \frac{p_2(\theta_{t'}) - p_2(\theta_t)}{p_2(\theta_t)}, \text{ if } \theta_{t'} < \theta_t \\
\frac{p_1(\theta_{t'}) - p_1(\theta_t)}{p_1(\theta_t)} &> \frac{p_2(\theta_{t'}) - p_2(\theta_t)}{p_2(\theta_t)}, \text{ if } \theta_{t'} > \theta_t.
\end{aligned}
$$

Under these conditions, when type $\theta_{t'}$ reports $\theta_t < \theta_{t'}$, it faces a lottery with lower expected value than if it were to tell the truth because, as compared to type θ_t, it is more likely to draw signal 2 and less likely to draw signal 1. Conversely, when type $\theta_{t'}$ reports $\theta_t > \theta_{t'}$, it faces a lottery with higher expected value because, as compared to type θ_t, it is less likely to draw signal 2 and more likely to draw signal 1. In addition to the lottery, the payoff of type $\theta_{t'}$

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[8]In Appendix A we show that, as long as (1) holds, $\mathbf{p}(\theta_1)$ and $\mathbf{p}(\theta_T)$ do not lie in the convex hull generated by the probability vectors of the other types. Moreover, there exist vectors $\mathbf{p}(\theta_t)$, $t \neq 1, T$, which lie in the convex hull generated by the probability vectors of the other types and do not violate (1).

reporting θ_t includes the difference between fake and real cost. Accordingly, the payoff of type $\theta_{t'}$ when reporting θ_t is given by:

$$\pi_2\left(\theta_t\right) p_2\left(\theta_t\right) \left(\frac{p_2(\theta_{t'})}{p_2\left(\theta_t\right)} - \frac{p_1(\theta_{t'})}{p_1\left(\theta_t\right)}\right) + C\left(q\left(\theta_t\right), \theta_t\right) - C\left(q\left(\theta_t\right), \theta_{t'}\right).$$

By looking at this payoff, we can asses what incentivizes type $\theta_{t'}$ to report θ_t. A type $\theta_{t'}$ that reports a higher type θ_t loses in terms of lottery but gains on the fixed payment obtained from P because, in that case, $C\left(q\left(\theta_t\right), \theta_{t'}\right) > C\left(q\left(\theta_{t'}\right), \theta_{t'}\right)$. On the opposite, a type $\theta_{t'}$ that reports a lower type θ_t loses on the fixed payment but gains in terms of lottery. Taking this into account, one can identify the requirements that the profit targeted to type θ_t for some signal realization must satisfy for the report θ_t to be unattractive to both lower and higher types. In particular, the profits assigned to the different types when signal 2 is realized must satisfy the following conditions:

$$\pi_2\left(\theta_1\right) \geq -\frac{C\left(q\left(\theta_1\right), \theta_{t'}\right) - C\left(q\left(\theta_1\right), \theta_1\right)}{p_2(\theta_1)\left(\frac{p_1(\theta_{t'})}{p_1(\theta_1)} - \frac{p_2(\theta_{t'})}{p_2(\theta_1)}\right)}, \ t' = 2, 3, \tag{13}$$

$$-\frac{C\left(q\left(\theta_2\right), \theta_3\right) - C\left(q\left(\theta_2\right), \theta_2\right)}{p_2(\theta_2)\left(\frac{p_1(\theta_3)}{p_1(\theta_2)} - \frac{p_2(\theta_3)}{p_2(\theta_2)}\right)} \leq \pi_2\left(\theta_2\right) \leq -\frac{C\left(q\left(\theta_2\right), \theta_2\right) - C\left(q\left(\theta_2\right), \theta_1\right)}{p_2(\theta_2)\left(\frac{p_2(\theta_1)}{p_2(\theta_2)} - \frac{p_1(\theta_1)}{p_1(\theta_2)}\right)} \tag{14}$$

and

$$\pi_2\left(\theta_3\right) \leq -\frac{C\left(q\left(\theta_3\right), \theta_3\right) - C\left(q\left(\theta_3\right), \theta_{t'}\right)}{p_2(\theta_3)\left(\frac{p_2(\theta_{t'})}{p_2(\theta_3)} - \frac{p_1(\theta_{t'})}{p_1(\theta_3)}\right)}, \ t' = 1, 2. \tag{15}$$

We see that, whereas it is easy to design lotteries such that none of the extreme types represents an attractive report for the other types, it is possible to design a lottery with such features for the intermediate type if and only if:

$$\frac{C\left(q\left(\theta_2\right), \theta_2\right) - C\left(q\left(\theta_2\right), \theta_1\right)}{C\left(q\left(\theta_2\right), \theta_3\right) - C\left(q\left(\theta_2\right), \theta_2\right)} \leq \frac{\frac{p_1(\theta_2) - p_1(\theta_1)}{p_1(\theta_2)} - \frac{p_2(\theta_2) - p_2(\theta_1)}{p_2(\theta_2)}}{\frac{p_1(\theta_3) - p_1(\theta_2)}{p_1(\theta_2)} - \frac{p_2(\theta_3) - p_2(\theta_2)}{p_2(\theta_2)}}. \tag{16}$$

The fact that type θ_1 gains on the fixed payment and loses on the lottery, whereas the converse occurs for type θ_3, involves that there exist values of $\pi_2\left(\theta_2\right)$ such that both types θ_1 and θ_3 are discouraged from claiming θ_2 if and only if the ratio between the gain to type θ_1 and the loss to type θ_3 in terms of fixed payment does not exceed the ratio between the loss to type θ_1 and the gain to type θ_3 in terms of lottery, as (16) shows. As the signal is binary, $p_2\left(\cdot\right) = 1 - p_1(\cdot)$ and one can rewrite (16) as (7), *i.e.* as the condition in Corollary 1.5 of RS. Recalling the explanation of condition (7), the gain/loss ratio in terms of fixed payment does not exceed the loss/gain ratio in terms of lottery if and only if the cost is less concave (more convex) than the conditional probability of signal 1.[9]

Next consider the issue of limited liability. The compensation scheme must be designed so

---

[9] We formulate the condition identified by RS as (7), rather than as the equivalent condition (16), because this is useful to prepare the reader to the subsequent analysis with more than two signals.

as to reconcile the attempt to prevent the agent from overstating type with the need to satisfy (LL). That is, (LIC) should not conflict with (LL). No conflict arises, indeed, if it is possible to set $\pi_2(\theta_2)$ and $\pi_2(\theta_3)$ sufficiently low to discourage lower types from over-reporting without yet exposing the agent to excessively high punishments. That is, it must be the case that:

$$(C(q(\theta_3),\theta_3) - C(q(\theta_3),\theta_{t'}))\frac{p_1(\theta_3)}{p_1(\theta_3) - p_1(\theta_{t'})} \leq L, \; t' = 1,2 \tag{17}$$

$$(C(q(\theta_2),\theta_2) - C(q(\theta_2),\theta_1))\frac{p_1(\theta_2)}{p_1(\theta_2) - p_1(\theta_1)} \leq L. \tag{18}$$

These two conditions are the counterpart of (10) in a setting with three types and two signals.

Overall, first best is at reach only if (16) (or, equivalently, (7)) holds jointly with (17) and (18). As shown in Appendix B.2, (16) also implies that the extreme types are more attracted by adjacent than non-adjacent types. Intuitively, because the lotteries that types θ_1 and θ_3 are faced with if announcing θ_2 are not too extreme when (16) holds, those types will prefer the claim θ_2 to the claim θ_3 and θ_1, respectively. The benefit of this is that both upward and downward incentive constraints must be verified locally only. Therefore, together with (17) and (18), (16) is also sufficient for first-best implementation.

3.2.2 Three signals

We now take $N = \{1,2,3\}$. Proceeding as above, we assess that, for the first-best allocation to be effected in an incentive-compatible manner, the following conditions should hold:

$$\pi_3(\theta_t) \leq -\frac{C(q(\theta_t),\theta_t) - C(q(\theta_t),\theta_{t'}) + \pi_2(\theta_t)p_2(\theta_t)\left(\frac{p_2(\theta_{t'})}{p_2(\theta_t)} - \frac{p_1(\theta_{t'})}{p_1(\theta_t)}\right)}{p_3(\theta_t)\left(\frac{p_3(\theta_{t'})}{p_3(\theta_t)} - \frac{p_1(\theta_{t'})}{p_1(\theta_t)}\right)}, \; \forall \theta_{t'} < \theta_t \tag{19}$$

$$\pi_3(\theta_t) \geq -\frac{C(q(\theta_t),\theta_{t'}) - C(q(\theta_t),\theta_t) + \pi_2(\theta_t)p_2(\theta_t)\left(\frac{p_1(\theta_{t'})}{p_1(\theta_t)} - \frac{p_2(\theta_{t'})}{p_2(\theta_t)}\right)}{p_3(\theta_t)\left(\frac{p_1(\theta_{t'})}{p_1(\theta_t)} - \frac{p_3(\theta_{t'})}{p_3(\theta_t)}\right)}, \; \forall \theta_{t'} > \theta_t. \tag{20}$$

Also in this setting it would not be an issue to construct lotteries such that the extreme types θ_1 and θ_3 do not represent attractive lies for any other type, whereas difficulties might arise with the intermediate type θ_2. When the compensation to the agent can be conditioned on three signals, rather than only two, the lottery admits one more level of profit. This provides P with an additional instrument to lessen the conflict between incentive constraints. The necessary condition (16) is replaced by:

$$\frac{C(q(\theta_2),\theta_2) - C(q(\theta_2),\theta_1)}{\frac{p_3(\theta_1)}{p_3(\theta_2)} - \frac{p_1(\theta_1)}{p_1(\theta_2)}} - \frac{C(q(\theta_2),\theta_3) - C(q(\theta_2),\theta_2)}{\frac{p_1(\theta_3)}{p_1(\theta_2)} - \frac{p_3(\theta_3)}{p_3(\theta_2)}} \tag{21}$$

$$\leq \pi_2(\theta_2)p_2(\theta_2)\left(\frac{\frac{p_1(\theta_3)}{p_1(\theta_2)} - \frac{p_2(\theta_3)}{p_2(\theta_2)}}{\frac{p_1(\theta_3)}{p_1(\theta_2)} - \frac{p_3(\theta_3)}{p_3(\theta_2)}} - \frac{\frac{p_2(\theta_1)}{p_2(\theta_2)} - \frac{p_1(\theta_1)}{p_1(\theta_2)}}{\frac{p_3(\theta_1)}{p_3(\theta_2)} - \frac{p_1(\theta_1)}{p_1(\theta_2)}}\right).$$

Assuming that the difference in brackets in the right-hand side is negative, (21) is most relaxed when $\pi_2(\theta_2)$ is decreased to the minimum: $\pi_2(\theta_2) = -L$. If the equality $\pi_3(\theta_2) = \pi_2(\theta_2)$ is imposed, then the necessary condition is again (16), and it can then be impossible to decrease $\pi_2(\theta_2)$ to $-L$. To see this, replace $\pi_2(\theta_t) = -L$ in (20) for the generic type θ_t and rearrange to obtain:

$$\pi_3(\theta_t) \geq -\frac{[C(q(\theta_t),\theta_{t'}) - C(q(\theta_t),\theta_t)] - L\frac{p_1(\theta_{t'})-p_1(\theta_t)}{p_1(\theta_t)}}{p_3(\theta_t)\left(\frac{p_1(\theta_{t'})}{p_1(\theta_t)} - \frac{p_3(\theta_{t'})}{p_3(\theta_t)}\right)} - L.$$

This shows that, if L is sufficiently high for (17) (or (18)) to hold strictly, then it must be the case that $\pi_3(\theta_t) > -L$. Therefore, the conflict between incentives is weakest when the profit associated with signal 2 is different from that associated with signal 3. Turning back to type θ_2, the condition under which the term that multiplies $\pi_2(\theta_2)$ is negative in (21), and hence it is optimal to set $\pi_2(\theta_2) = -L$, is given by:

$$\frac{\frac{p_1(\theta_2)-p_1(\theta_1)}{p_1(\theta_2)} - \frac{p_2(\theta_2)-p_2(\theta_1)}{p_2(\theta_2)}}{\frac{p_1(\theta_3)-p_1(\theta_2)}{p_1(\theta_2)} - \frac{p_2(\theta_3)-p_2(\theta_2)}{p_2(\theta_2)}} > \frac{\frac{p_1(\theta_2)-p_1(\theta_1)}{p_1(\theta_2)} - \frac{p_3(\theta_2)-p_3(\theta_1)}{p_3(\theta_2)}}{\frac{p_1(\theta_3)-p_1(\theta_2)}{p_1(\theta_2)} - \frac{p_3(\theta_3)-p_3(\theta_2)}{p_3(\theta_2)}}. \tag{22}$$

Notice that the left-hand side of (22) replicates the right-hand side of (16). Furthermore, the left- and the right-hand side of (22) are just the same except that the likelihood of signal 2 in the former is replaced by that of signal 3 in the latter. These observations are useful to interpret (22). If $\pi_2(\theta_2)$ is decreased, then by announcing θ_2 type θ_1 loses and type θ_3 gains in terms of lottery. As long as the ratio between such loss and gain exceeds the ratio that would result from a decrease in $\pi_3(\theta_2)$ rather than in $\pi_2(\theta_2)$, the best strategy is to set $\pi_2(\theta_2) = -L$. Obviously, in the converse case, the best strategy would be to set $\pi_3(\theta_2) = -L$, instead. In any case, the lottery that is most likely to implement first best departs from that pinned down by GBS, which is such that $\pi_2(\theta_2) = \pi_3(\theta_2)$.

As in situations with a binary signal, provided that (21) holds, and hence one can construct a lottery that is locally incentive-compatible, it is further necessary to take care of the possible incentives to mimic non-adjacent types. However, as compared to situations with a binary signal, it is now less obvious that the incentives to mimic adjacent types prevail on those to mimic non-adjacent types. By extending our investigation to a continuum of types here below, we show that, in fact, adjacent types are more attractive reports also in settings with more than two types, which simplifies the analysis significantly.

3.3 A continuum of types and a finite number of signals

Consider a continuum of types and $n \geq 2$. In this framework, it is useful to state local and global incentive constraints separately, as follows (see Appendix C for the derivation):

$$C_\theta \left(q \left(\theta \right), \theta \right) = \sum_{s=1}^{n} \pi_s \left(\theta \right) p'_s \left(\theta \right), \ \forall \theta \in \Theta \tag{LIC}$$

$$C(q(\theta), \theta) - C(q(\theta), \theta') \leq \sum_{s=1}^{n} \pi_s(\theta)(p_s(\theta) - p_s(\theta')), \ \forall \theta', \theta \in \Theta. \tag{GIC}$$

These constraints ensure that report θ is not attractive to any type $\theta' \neq \theta$, and hence it will be chosen by type θ only, when all surplus is extracted from the agent. According to (LIC), when θ' is in a neighborhood of θ, any benefit from a lie is eliminated if P designs profits for type θ such that the marginal change that type θ' would face in the lottery part of its compensation after claiming θ ($\sum_{s=1}^{n} \pi_s \left(\theta \right) p'_s \left(\theta \right)$) is just as great as the marginal change it would face in the fixed payment ($C_\theta \left(q \left(\theta \right), \theta \right)$). Any deviation away from this rule would make the lie worth for some neighboring types: for higher types, if the marginal change in the lottery is greater than the marginal change in the fixed payment; for lower types, if the converse occurs. In the same vein, (GIC) suggests that θ is not an attractive report for a non-neighboring type θ' if the fixed gain from cost exaggeration ($C(q(\theta), \theta) - C(q(\theta), \theta')$) is lower than the lottery loss ($\sum_{s=1}^{n} \pi_s(\theta)(p_s((\theta) - p_s(\theta')))$), when $\theta' < \theta$; and if the fixed loss from cost understatement ($C(q(\theta), \theta') - C(q(\theta), \theta)$) exceeds the lottery gain ($\sum_{s=1}^{n} \pi_s(\theta)(p_s(\theta') - p_s(\theta))$), when $\theta' > \theta$ instead.

Next consider (LL), which must hold jointly with (LIC) and (GIC). To investigate the impact of (LL), it is first useful to rewrite (LIC) as follows:

$$\pi_n \left(\theta \right) = \frac{C_\theta(q(\theta), \theta)) + \sum_{s \neq 1, n} \pi_s(\theta) p_s(\theta) \left(\frac{p'_1(\theta)}{p_1(\theta)} - \frac{p'_s(\theta)}{p_s(\theta)} \right)}{-p_n(\theta) \left(\frac{p'_1(\theta)}{p_1(\theta)} - \frac{p'_n(\theta)}{p_n(\theta)} \right)}. \tag{23}$$

Given the profits associated with the first $n-1$ signals in the lottery assigned to type θ, it is impossible to prevent all neighboring types from reporting θ unless $\pi_n(\theta)$ is set as in (23). Next recall from GBS that the conflict between local incentive-compatibility and limited liability is most likely eliminated if the principal adopts the lottery $\boldsymbol{\pi}^1(\theta) \ \forall \theta$. That lottery is obtained by replacing $\pi_s(\theta) = \pi_n(\theta)$ into (23) and saturating (PC). One can see the result of GBS as an implication of the following lemma (the proof is in Appendix D).[10]

Lemma 1 *Take $n \geq 3$, $\boldsymbol{\pi}(\theta) \in \Pi(\theta) \ \forall \theta \in \Theta$, and any triplet of signals $\{i, j, k\}$ such that:*

$$\frac{p'_i(\theta)}{p_i(\theta)} > \frac{p'_j(\theta)}{p_j(\theta)} > \frac{p'_k(\theta)}{p_k(\theta)}, \ \forall \theta \in \Theta. \tag{24}$$

---

[10]To prove that punishments should all be equal for (LIC) to hold jointly with (LL), GBS consider three different profits and show that (LL) is most likely satisfied if the two lowest profits out of the three are adjusted to become equal.

Given $\pi_s(\theta) \in \boldsymbol{\pi}(\theta)$, $\forall s \notin \{i,j,k\}$, if a change is induced in $\pi_i(\theta)$, then the new lottery belongs to $\boldsymbol{\Pi}(\theta)$ only if changes are also induced in $\pi_j(\theta)$ and $\pi_k(\theta)$, in opposite directions.

Switching from $\boldsymbol{\pi}^1(\theta)$ to a different lottery also belonging to $\boldsymbol{\Pi}(\theta)$ requires changing two profits $\pi_j(\theta)$ and $\pi_k(\theta)$ in opposite directions; otherwise (4) would be violated. As one profit is decreased, (LL) is tightened. Recall from the discrete-type case that limited liability may conflict with local incentive-compatibility only for lower-order types, which might be willing to overstate information. Provided that the probability of signal i changes at a greater rate than that of signal j as type increases from some θ^- to $\theta > \theta^-$, a lower value of $\pi_j(\theta)$ can be compensated with a higher value of $\pi_i(\theta)$, without changing any other profit, in such a way that type θ still faces a lottery with the same expected value but type θ^- is now less attracted by the lottery targeted to θ. Analogous outcome could alternatively be induced by increasing either $\pi_i(\theta)$ or $\pi_j(\theta)$, whereas $\pi_k(\theta)$ is decreased. Notice however that any such pair of profit changes would make a report θ more attractive to some higher type θ^+.[11] As there is a continuum of types, P cannot induce changes in only two of the profits designed for a certain type and still extract surplus from all types. This explains why P must induce changes in three profits when switching to a new lottery which also yields full surplus extraction. In so doing, $\pi_j(\theta)$ and $\pi_k(\theta)$ must change in opposite directions. Remarkably, (24) imposes no restrictions on the curvature of the probability functions so that $p_i(\cdot)$, say, can be less or more concave/convex than $p_j(\cdot)$.

Consider now the potential conflict between (GIC) and (LL). Recall that Corollary 1.5 in RS shows that, if $\exists i \in N$ such that the cost function is less concave in type than the probability function of signal i ((7) is satisfied), then first best is attained by using the lottery $\boldsymbol{\pi}^i(\theta)$ $\forall \theta$. In line with the result of RS, GBS rule out any difficulty with (GIC) by assuming that the cost function is convex in type and that the likelihood function of signal $i = 1$ is concave in type.

The main question to our study is whether using the lottery $\boldsymbol{\pi}^1(\theta)$ is still the optimal strategy when the cost function is more concave than the probability function of the reward signal and (LL) is not binding when that lottery is adopted. More specifically, we aim at understanding whether and how P could take advantage of the liability slack to make the contract incentive-compatible. To that end, it is useful to decompose (GIC) into two distinct conditions whereby some given report is unattractive to lower and higher types.

Lemma 2 *Given (PC) and (LIC), (GIC) is rewritten as the following pair of conditions for any given θ:*

$$
C_\theta(q(\theta),\theta) \geq \left(\frac{p_1'(\theta)}{p_1(\theta)} - \frac{p_n'(\theta)}{p_n(\theta)}\right)\left[\frac{C(q(\theta),\theta) - C(q(\theta),\theta^-)}{\frac{p_n(\theta^-)}{p_n(\theta)} - \frac{p_1(\theta^-)}{p_1(\theta)}}\right. \tag{25}
$$
$$
\left. + \sum_{s\neq 1,n}\pi_s(\theta)p_s(\theta)\left(\frac{\frac{p_1(\theta^-)}{p_1(\theta)} - \frac{p_s(\theta^-)}{p_s(\theta)}}{\frac{p_1(\theta^-)}{p_1(\theta)} - \frac{p_n(\theta^-)}{p_n(\theta)}} - \frac{\frac{p_1'(\theta)}{p_1(\theta)} - \frac{p_S'(\theta)}{p_S(\theta)}}{\frac{p_1'(\theta)}{p_1(\theta)} - \frac{p_n'(\theta)}{p_n(\theta)}}\right)\right], \forall \theta^- < \theta,
$$

---

[11]We let θ^- and θ^+ denote types respectively below and above θ, but not necessarily limit values around θ.

and

$$C_\theta\left(q\left(\theta\right),\theta\right) \leq \left(\frac{p_1'\left(\theta\right)}{p_1\left(\theta\right)} - \frac{p_n'\left(\theta\right)}{p_n\left(\theta\right)}\right)\left[\frac{C\left(q\left(\theta\right),\theta^+\right) - C\left(q\left(\theta\right),\theta\right)}{\frac{p_1\left(\theta^+\right)}{p_1\left(\theta\right)} - \frac{p_n\left(\theta^+\right)}{p_n\left(\theta\right)}}\right.$$
$$\left. + \sum_{s\neq 1,n} \pi_s\left(\theta\right)p_s\left(\theta\right)\left(\frac{\frac{p_1\left(\theta^+\right)}{p_1\left(\theta\right)} - \frac{p_s\left(\theta^+\right)}{p_s\left(\theta\right)}}{\frac{p_1\left(\theta^+\right)}{p_1\left(\theta\right)} - \frac{p_n\left(\theta^+\right)}{p_n\left(\theta\right)}} - \frac{\frac{p_1'\left(\theta\right)}{p_1\left(\theta\right)} - \frac{p_s'\left(\theta\right)}{p_s\left(\theta\right)}}{\frac{p_1'\left(\theta\right)}{p_1\left(\theta\right)} - \frac{p_n'\left(\theta\right)}{p_n\left(\theta\right)}}\right)\right], \;\; \forall \theta^+ > \theta. \tag{26}$$

Similarly to the discrete-type case, and for the reasons there explained, there is a potential conflict between (25) and (26). To avoid rise of the conflict, it is necessary to have the following condition satisfied for each possible triplet $\left\{\theta^-, \theta, \theta^+\right\}$:

$$\frac{C\left(q\left(\theta\right),\theta\right) - C\left(q\left(\theta\right),\theta^-\right)}{\frac{p_n\left(\theta^-\right)}{p_n\left(\theta\right)} - \frac{p_1\left(\theta^-\right)}{p_1\left(\theta\right)}} - \frac{C\left(q\left(\theta\right),\theta^+\right) - C\left(q\left(\theta\right),\theta\right)}{\frac{p_1\left(\theta^+\right)}{p_1\left(\theta\right)} - \frac{p_n\left(\theta^+\right)}{p_n\left(\theta\right)}} \tag{27}$$
$$\leq \sum_{s\neq 1,n} \pi_s\left(\theta\right)p_s\left(\theta\right)\left(\frac{\frac{p_1\left(\theta^+\right)}{p_1\left(\theta\right)} - \frac{p_s\left(\theta^+\right)}{p_s\left(\theta\right)}}{\frac{p_1\left(\theta^+\right)}{p_1\left(\theta\right)} - \frac{p_n\left(\theta^+\right)}{p_n\left(\theta\right)}} - \frac{\frac{p_1\left(\theta^-\right)}{p_1\left(\theta\right)} - \frac{p_s\left(\theta^-\right)}{p_s\left(\theta\right)}}{\frac{p_1\left(\theta^-\right)}{p_1\left(\theta\right)} - \frac{p_n\left(\theta^-\right)}{p_n\left(\theta\right)}}\right).$$

Therefore, one needs first to check whether, for each possible report θ, there exists a lottery such that (27) holds without violating (LL). Once this is ascertained, one further needs to verify that such a lottery satisfies (25) and (26). As this is required for all possible pairs $\left\{\theta^-, \theta^+\right\}$, the analysis looks complex overall. The problem is tractable, in fact, thanks to the following result.

Lemma 3 (27) *is necessary and sufficient for* (25) *and* (26) *to hold.*

Once it is established that it suffices to check (27) to verify (25) and (26), it is possible to pin down the optimal incentive scheme according to the properties of the cost and the likelihood functions. To that end, it is useful to define:

$$\rho_s\left(\theta',\theta\right) \equiv \frac{p_s\left(\theta'\right) + \left(\theta - \theta'\right)p_s'\left(\theta'\right)}{p_s\left(\theta\right)}, \;\; \forall \theta' \neq \theta \in \Theta, \; \forall s \in N,$$

where $\rho_s\left(\theta',\theta\right) = 1$ if $p_s\left(\cdot\right)$ is linear, $\rho_s\left(\theta',\theta\right) < 1$ if $p_s\left(\cdot\right)$ is strictly convex, and $\rho_s\left(\theta',\theta\right) > 1$ if $p_s\left(\cdot\right)$ is strictly concave. The more that $\rho_s\left(\theta',\theta\right)$ diverges from 1, the higher that the degree of convexity/concavity of $p_s\left(\cdot\right)$ is $\forall \theta' \neq \theta$. Hence, the magnitude of $\rho_s\left(\cdot,\cdot\right)$ is a measure of the curvature of the probability function of signal s. Using this definition, one can show that if

$$\frac{\rho_s\left(\theta',\theta\right) - \rho_1\left(\theta',\theta\right)}{\left|\frac{p_1\left(\theta'\right)}{p_1\left(\theta\right)} - \frac{p_s\left(\theta'\right)}{p_s\left(\theta\right)}\right|} < \frac{\rho_n\left(\theta',\theta\right) - \rho_1\left(\theta',\theta\right)}{\left|\frac{p_1\left(\theta'\right)}{p_1\left(\theta\right)} - \frac{p_n\left(\theta'\right)}{p_n\left(\theta\right)}\right|}, \tag{28}$$

then the term in brackets in the right-hand side of (27) is negative $\forall \theta$ such that $\theta^- \leq \theta \leq \theta^+$, with at least one of these inequalities holding strictly (see the proof of Proposition 1 below in Appendix G.1). Assuming that this is true, the lottery that is most likely to implement first

best, denoted $\boldsymbol{\pi}^*(\theta)$, includes the following list of profits $\forall \theta \in \Theta$:

$$\pi_1^*(\theta) = \frac{C_\theta(q(\theta),\theta) - L\frac{p_n'(\theta)}{p_n(\theta)}}{p_1(\theta)\left(\frac{p_1'(\theta)}{p_1(\theta)} - \frac{p_n'(\theta)}{p_n(\theta)}\right)} - L \tag{29}$$

$$\pi_n^*(\theta) = \frac{L\frac{p_1'(\theta)}{p_1(\theta)} - C_\theta(q(\theta),\theta)}{p_n(\theta)\left(\frac{p_1'(\theta)}{p_1(\theta)} - \frac{p_n'(\theta)}{p_n(\theta)}\right)} - L \tag{30}$$

$$\pi_s^*(\theta) = -L, \ \forall s \neq 1, n. \tag{31}$$

Proposition 1 *Assume that $n \geq 3$ and that (28) holds. Then, first best is implemented if and only if either:*

$$\frac{C(q(\theta),\theta) - C(q(\theta),\theta^-)}{C(q(\theta),\theta^+) - C(q(\theta),\theta)} \leq \frac{p_1(\theta) - p_1(\theta^-)}{p_1(\theta^+) - p_1(\theta)}, \ \forall \theta, \theta^-, \theta^+ \in \Theta, \ \theta^- < \theta < \theta^+ \tag{32}$$

and

$$L \geq \left(C(q(\theta),\theta) - C(q(\theta),\theta^-)\right)\frac{p_1(\theta)}{p_1(\theta) - p_1(\theta^-)}, \ \forall \theta^-, \theta \in \Theta, \ \theta^- < \theta \tag{33}$$

or (32) is violated and:

$$L \geq \frac{\frac{C(q(\theta),\theta) - C(q(\theta),\theta^-)}{\frac{p_n(\theta^-)}{p_n(\theta)} - \frac{p_1(\theta^-)}{p_1(\theta)}} - \frac{C(q(\theta),\theta^+) - C(q(\theta),\theta)}{\frac{p_1(\theta^+)}{p_1(\theta)} - \frac{p_n(\theta^+)}{p_n(\theta)}}}{-\sum_{s \neq 1,n} p_s(\theta)\left(\frac{\frac{p_1(\theta^+)}{p_1(\theta)} - \frac{p_s(\theta^+)}{p_s(\theta)}}{\frac{p_1(\theta^+)}{p_1(\theta)} - \frac{p_n(\theta^+)}{p_n(\theta)}} - \frac{\frac{p_1(\theta^-)}{p_1(\theta)} - \frac{p_s(\theta^-)}{p_s(\theta)}}{\frac{p_1(\theta^-)}{p_1(\theta)} - \frac{p_n(\theta^-)}{p_n(\theta)}}\right)}, \ \forall \theta, \theta^-, \theta^+ \in \Theta, \ \theta^- < \theta < \theta^+. \tag{34}$$

This proposition extends Proposition 2 of GBS, where condition (33) is required under the assumption that the cost is convex in type, to the case where the cost is possibly concave in type, as captured by condition (32) of Corollary 1.5 of RS, and, more importantly, to the case where (32) does not hold jointly with (33) but first-best is still implemented.[12] This result is useful in that it draws a single condition to be satisfied for first-best implementation when (32) does not hold, a condition which depends on how liable the agent is and on the properties of the cost and the likelihood functions. To interpret the result, it is first necessary to recall that it was obtained by identifying the lottery $\boldsymbol{\pi}^*(\theta)$ as being most likely to yield the first-best outcome. It is then useful to go through the following corollaries.

Corollary 1 $\pi_1^*(\theta) > \pi_1^1(\theta)$, $\pi_n^*(\theta) > \pi_n^1(\theta)$ and $\pi_s^*(\theta) < \pi_s^1(\theta)$, $\forall s \neq 1, n, \forall \theta$.

This corollary evidences in which way $\boldsymbol{\pi}^*(\theta)$ departs from the lottery pinned down by GBS. When the cost and the probability functions display the properties stated in Proposition 1, P should rely on Lemma 1 and proceed as follows. Starting from $\boldsymbol{\pi}^1(\theta)$, she should raise the profit associated with signal n, in addition to that associated with signal 1, and decrease the

---

profits associated with all the other signals. According to Lemma 1, P gains flexibility when switching from $\boldsymbol{\pi}^1(\theta)$ to a new lottery in which the profit associated with signal 1 is raised and opposite changes are induced in the profits associated with two other signals. As explained in the case with discrete types, it is convenient to increase the profit of type θ in state 1 and decrease it in some state $s \neq 1$ because type θ^- is then led to bear a greater loss when reporting θ. This is because $\frac{p_1'(\theta)}{p_1(\theta)} > \frac{p_s'(\theta)}{p_s(\theta)}$, $\forall s \neq 1$, involving that type θ^- will obtain less with a signal that it is more likely to draw and more with a signal that it is less likely to draw. This process can be replicated for signal 1 and other $n-2$ signals with which profits higher than $-L$ are initially associated. On the other hand, for one signal realization the profit must be increased in order to weaken the incentive of type θ^- to exaggerate information. The remaining question is thus for which signal realization, beside 1, the profit should be increased and for which ones it should be decreased instead. Corollary 1 identifies those signals.

Corollary 2 (25) *is relaxed and* (26) *is tightened when* $\boldsymbol{\pi}^*(\theta)$ *replaces* $\boldsymbol{\pi}^1(\theta)$.

This result formalizes the impossibility of lessening the global incentives both to overstate and to understate information by switching from one lottery to another in $\Pi(\theta)$ $\forall\theta$. However, provided that (28) holds, when replacing $\boldsymbol{\pi}^1(\theta)$ with $\boldsymbol{\pi}^*(\theta)$ the positive effect of type θ^- becoming less eager to claim θ prevails on the negative effect of type θ^+ becoming more eager to do so. Indeed, under (28), one has:

$$\frac{\frac{p_1(\theta^+)}{p_1(\theta)} - \frac{p_s(\theta^+)}{p_s(\theta)}}{\frac{p_1(\theta^+)}{p_1(\theta)} - \frac{p_n(\theta^+)}{p_n(\theta)}} < \frac{\frac{p_1(\theta^-)}{p_1(\theta)} - \frac{p_s(\theta^-)}{p_s(\theta)}}{\frac{p_1(\theta^-)}{p_1(\theta)} - \frac{p_n(\theta^-)}{p_n(\theta)}}, \quad \forall \theta^-, \theta, \theta^+ \in \Theta : \theta \in (\theta^-, \theta^+), \tag{35}$$

which is the counterpart of (22) in a setting with more than three types. Under (35), it is easier to lessen the conflict between the incentive constraints "from below" and "from above" if the profits of type θ are decreased to $-L$ for all signals but 1 and n, rather than for all signals but n only. Remarkably, when (GIC) is not a concern as in the setting considered by GBS, it suffices to refer to the rate of change of the conditional probability to determine the lottery that is most likely to eliminate the tension between local incentive-compatibility and limited liability. However, this is no longer the only requirement to be met in terms of probabilities as it comes to the incentive scheme that makes the tension between (GIC) and (LL) weakest. The curvature of the function $p(\cdot)$ becomes important as well because the potential gains and losses from the different lies depend on how the probabilities of the signals vary with type. The next corollary lists the necessary and sufficient conditions for (28) to hold, and hence for (27) to be weakest.

Corollary 3 *For* (28) *to hold* $\forall s \neq 1, n$:
    it is necessary that $\rho_s(\theta', \theta) < \max\{\rho_1(\theta', \theta), \rho_n(\theta', \theta)\}$ *and sufficient that either*

$$\rho_s(\theta', \theta) < \rho_1(\theta', \theta) < \rho_n(\theta', \theta)$$

or

$$\rho_n\left(\theta',\theta\right) < \rho_s\left(\theta',\theta\right) < \rho_1\left(\theta',\theta\right);$$

it is necessary and sufficient that $\rho_n\left(\theta',\theta\right) - \rho_s\left(\theta',\theta\right)$ *be "sufficiently large" when*

$$\rho_1\left(\theta',\theta\right) < \rho_s\left(\theta',\theta\right) < \rho_n\left(\theta',\theta\right),$$

and that $\rho_n\left(\theta',\theta\right) - \rho_s\left(\theta',\theta\right)$ *be "sufficiently small" when*

$$\rho_s\left(\theta',\theta\right) < \rho_n\left(\theta',\theta\right) < \rho_1\left(\theta',\theta\right).$$

Intuitively, because any decrease in $\pi_s\left(\theta\right)$ is compensated with an increase in both $\pi_1\left(\theta\right)$ and $\pi_n\left(\theta\right)$ (recall Lemma 1), the lottery $\boldsymbol{\pi}^*\left(\theta\right)$ cannot be employed unless at least one between $p_1\left(\cdot\right)$ and $p_n\left(\cdot\right)$ is less convex / more concave than the conditional probability of any other signal. If this is not the case, then incentives to understate information are too strong for $\boldsymbol{\pi}^*\left(\theta\right)$ to weaken (27). Specifically, (26) is tightened more than (25) is relaxed (recall Corollary 2). The remaining conditions listed in Corollary 3 are sufficient conditions on the degree of concavity/convexity of the likelihood functions for (28) to hold.

In substance, as long as (LL) does not bind in $\boldsymbol{\pi}^1\left(\theta\right)$ at least for some θ, the gain that P obtains by moving away from that lottery in such a way as to take advantage of the slack of (LL), resides in that global incentive-compatibility is reconciled with limited liability for a wider family of cost functions. That is, first best is at hand in a richer variety of contractual relationships. To see this, rewrite (34) as follows:

$$
\frac{C\left(q\left(\theta\right),\theta\right) - C\left(q\left(\theta\right),\theta^-\right)}{C\left(q\left(\theta\right),\theta^+\right) - C\left(q\left(\theta\right),\theta\right)}
\tag{36}
$$
$$
\leq \frac{p_1\left(\theta\right) - p_1\left(\theta^-\right)}{p_1\left(\theta^+\right) - p_1\left(\theta\right)} + \left(\frac{\frac{p_n\left(\theta^-\right)}{p_n\left(\theta\right)} - \frac{p_1\left(\theta^-\right)}{p_1\left(\theta\right)}}{\frac{p_1\left(\theta^+\right)}{p_1\left(\theta\right)} - \frac{p_n\left(\theta^+\right)}{p_n\left(\theta\right)}} - \frac{\frac{p_1\left(\theta\right) - p_1\left(\theta^-\right)}{p_1\left(\theta\right)}}{\frac{p_1\left(\theta^+\right) - p_1\left(\theta\right)}{p_1\left(\theta\right)}}\right)
$$
$$
- \frac{L\left(\frac{p_n\left(\theta^-\right)}{p_n\left(\theta\right)} - \frac{p_1\left(\theta^-\right)}{p_1\left(\theta\right)}\right)}{C\left(q\left(\theta\right),\theta^+\right) - C\left(q\left(\theta\right),\theta\right)} \sum_{s \neq 1,n} p_s\left(\theta\right) \left(\frac{\frac{p_1\left(\theta^+\right)}{p_1\left(\theta\right)} - \frac{p_s\left(\theta^+\right)}{p_s\left(\theta\right)}}{\frac{p_1\left(\theta^+\right)}{p_1\left(\theta\right)} - \frac{p_n\left(\theta^+\right)}{p_n\left(\theta\right)}} - \frac{\frac{p_1\left(\theta^-\right)}{p_1\left(\theta\right)} - \frac{p_s\left(\theta^-\right)}{p_s\left(\theta\right)}}{\frac{p_1\left(\theta^-\right)}{p_1\left(\theta\right)} - \frac{p_n\left(\theta^-\right)}{p_n\left(\theta\right)}}\right),
$$

and observe that the last two terms in the right-hand side of (36), which are both positive, do not appear in the right-hand side of (16).

Corollary 4 (34) *is weaker than* (16).

This involves that the restrictions on the cost function are weaker than the sufficient condition identified by RS. Hence, in situations in which the conditional probabilities satisfy the assumptions previously made, P attains incentive-compatibility under milder conditions by switching from $\boldsymbol{\pi}^1\left(\theta\right)$ to $\boldsymbol{\pi}^*\left(\theta\right)$, $\forall\theta \in \left(\underline{\theta},\overline{\theta}\right)$. In fact, $\boldsymbol{\pi}^*\left(\theta\right)$ is the lottery such that the restrictions on the cost function are weakest. Furthermore, this outcome is achieved only if the

extent of the liability is higher than required by GBS.

Corollary 5 (34) *implies* (33) *if and only if* (32) *is violated.*

There is a simple conclusion to be drawn from this result. P can shift from $\boldsymbol{\pi}^1(\theta)$ to $\boldsymbol{\pi}^*(\theta)$ as long as (33) is slack, and she can take advantage of that slackness to relax the incentive-compatibility constraints.

4 Second best

There are multiple possible departures from first best. One of them occurs when (32) is satisfied but (33) is not, which is the case GBS consider in their second-best analysis. In that case, (LIC) cannot hold together with (LL) unless P deviates from the first-best allocation. When it is (32) to be violated instead, one possibility is (28) not holding in Proposition 1 for at least one of the signals 1 and n, selected according to (1). However, intuition suggests that the lottery which is most likely to attain first best will then have similar characteristics to $\boldsymbol{\pi}^*(\theta)$, except that a pair of signals other than $\{1, n\}$ will be selected to satisfy (PC) and (LIC), involving that (34) will be tighter. A more interesting possibility to consider is that (34) does not hold, thus ruling out the second option in Proposition 1, which otherwise applies when (32) is violated. This is the case we now turn to explore. To that end, for simplicity, we take (34) to be violated by any triplet $\{\theta^-, \theta, \theta^+\}$ drawn from the feasible set. Notice that because (34) implies (33) (Corollary 5), our investigation will also include the case considered by GBS.

Resting on our first-best analysis, two questions arise naturally with regards to the case in which (32) is violated and L is not great enough for (34) to hold. First, one would like to know whether the optimal lottery still displays the feature that, for each type, all profits but those associated with signals 1 and n are equal. Second, one wonders whether there is any type to be conceded an information rent at the second-best optimum.

We hereafter develop the second-best analysis focusing, to begin with, on three types, and then on a continuum of types. We content ourselves with describing the main aspects of the analysis, providing intuition about results. To keep notation parsimonious, we go on denoting $q(\theta)$ the quantity of a generic type θ in the second-best setting, with the understanding that it no longer refers to the first-best production level. Accordingly, $\pi(\theta)$ will denote the profit assigned for the production of that quantity.

4.1 Three types

Consider again a setting with three types such that $\theta_1 < \theta_2 < \theta_3$ and assume that, as long as P insists on the first-best allocation, the incentive-compatibility constraints whereby the extreme types θ_1 and θ_3 are unwilling to claim θ_2 cannot be satisfied at once. The issue is then whether any of these types should be conceded an information rent to be motivated to tell the

truth, and whether or not it is possible to extract all surplus from type θ_2. For simplicity, we let the expected value of the lottery be $R(\theta_t) \equiv \sum_s \pi(\theta_t) p_s(\theta_t), \forall \theta_t \in \Theta_3$. The incentive constraints whereby types θ_1 and θ_3 are unwilling to claim θ_2 are given by:

$$\pi_n(\theta_2) p_n(\theta_2) \leq \frac{R(\theta_1) - \frac{p_1(\theta_1)}{p_1(\theta_2)} R(\theta_2)}{\frac{p_n(\theta_1)}{p_n(\theta_2)} - \frac{p_1(\theta_1)}{p_1(\theta_2)}} \tag{37}$$

$$- \frac{C(q(\theta_2), \theta_2) - C(q(\theta_2), \theta_1) + \sum_{s \neq 1, n} \pi_s(\theta_2) p_s(\theta_2) \left(\frac{p_s(\theta_1)}{p_s(\theta_2)} - \frac{p_1(\theta_1)}{p_1(\theta_2)} \right)}{\frac{p_n(\theta_1)}{p_n(\theta_2)} - \frac{p_1(\theta_1)}{p_1(\theta_2)}}$$

$$\pi_n(\theta_2) p_n(\theta_2) \geq \frac{\frac{p_1(\theta_3)}{p_1(\theta_2)} R(\theta_2) - R(\theta_3)}{\frac{p_1(\theta_3)}{p_1(\theta_2)} - \frac{p_n(\theta_3)}{p_n(\theta_2)}} \tag{38}$$

$$- \frac{C(q(\theta_2), \theta_3) - C(q(\theta_2), \theta_2) + \sum_{s \neq 1, n} \pi_s(\theta_2) p_s(\theta_2) \left(\frac{p_1(\theta_3)}{p_1(\theta_2)} - \frac{p_s(\theta_3)}{p_s(\theta_2)} \right)}{\frac{p_1(\theta_3)}{p_1(\theta_2)} - \frac{p_n(\theta_3)}{p_n(\theta_2)}}.$$

There other two adjacent incentive constraints to be considered are those whereby type θ_2 is willing to announce neither θ_3 nor θ_1 :

$$R(\theta_2) - \frac{p_1(\theta_2)}{p_1(\theta_3)} R(\theta_3) - [C(q(\theta_3), \theta_3) - C(q(\theta_3), \theta_2)] \tag{39}$$

$$- \sum_{s \neq 1} \pi_s(\theta_3) p_s(\theta_3) \left(\frac{p_s(\theta_2)}{p_s(\theta_3)} - \frac{p_1(\theta_2)}{p_1(\theta_3)} \right) \geq 0$$

$$R(\theta_2) - \frac{p_1(\theta_2)}{p_1(\theta_1)} R(\theta_1) - [C(q(\theta_1), \theta_1) - C(q(\theta_1), \theta_2)] \tag{40}$$

$$+ \sum_{s \neq 1} \pi_s(\theta_1) p_s(\theta_1) \left(\frac{p_1(\theta_2)}{p_1(\theta_1)} - \frac{p_s(\theta_2)}{p_s(\theta_1)} \right) \geq 0.$$

At optimum, both (37) and (38) are binding. This is because, given the rents designed for the three types, an increase in $\pi_n(\theta_2)$ (associated with a decrease in $\pi_1(\theta_2)$) strengthens the incentive of type θ_1 to claim θ_2 to the same extent that it lessens the incentive of type θ_3 to claim θ_2. In other words, the marginal cost of tightening (37) is exactly equal to the marginal benefit of relaxing (38). Moreover, the profits designed for type θ_2 are optimally decreased to $-L$ for all signal realizations but 1 and n. The reason is that, as in the first-best setting, this makes it least likely that (37) conflicts with (38). It follows that the second-best lottery is structured either as $\boldsymbol{\pi}^1(\theta)$ or as $\boldsymbol{\pi}^*(\theta)$, depending on whether or not (LL) is binding for the intermediate type when signal n is realized (this does not entail that quantities will be set at their first-best levels though). Interestingly, the value of L that separates the regime in which $\pi_n(\theta_2) = -L$ (and hence, the lottery is structured as $\boldsymbol{\pi}^1(\theta)$) from that in which $\pi_n(\theta_2) > -L$ (and hence, the lottery is structured as $\boldsymbol{\pi}^*(\theta)$) is also the value that separates the regime in which types θ_1 and θ_2 are assigned an information rent from that in which they are not. We present these two cases below.

4.1.1 Condition (33) is violated

Setting $\pi_s(\theta_2) = -L, \forall s \neq 1, n$ to reformulate (37) and (38) as:

$$
\begin{aligned}
R(\theta_1) \geq {}& \frac{p_1(\theta_1)}{p_1(\theta_2)} R(\theta_2) + C(q(\theta_2), \theta_2) - C(q(\theta_2), \theta_1) - L \sum_{s \neq 1, n} p_s(\theta_2) \left(\frac{p_s(\theta_1)}{p_s(\theta_2)} - \frac{p_1(\theta_1)}{p_1(\theta_2)} \right) \\
& + \pi_n(\theta_2) p_n(\theta_2) \left(\frac{p_n(\theta_1)}{p_n(\theta_2)} - \frac{p_1(\theta_1)}{p_1(\theta_2)} \right) \\
R(\theta_2) \geq {}& \frac{p_1(\theta_2)}{p_1(\theta_3)} \left\{ R(\theta_3) + C(q(\theta_2), \theta_3) - C(q(\theta_2), \theta_2) - L \sum_{s \neq 1, n} p_s(\theta_2) \left(\frac{p_1(\theta_3)}{p_1(\theta_2)} - \frac{p_s(\theta_3)}{p_s(\theta_2)} \right) \right. \\
& \left. + \pi_n(\theta_2) p_n(\theta_2) \left(\frac{p_1(\theta_3)}{p_1(\theta_2)} - \frac{p_n(\theta_3)}{p_n(\theta_2)} \right) \right\},
\end{aligned}
$$

we see that they are both weaker if type θ_2 is assigned the minimum profit of $-L$ also when signal n is realized. Besides, the latter constraint is further relaxed when $R(\theta_3)$ is downsized to zero. With these optimal values, (37) and (38) ultimately collapse onto (39) and (40), and they are all binding. The lottery has a similar structure to $\boldsymbol{\pi}^1(\theta)$, as we mentioned. To prevent cost exaggeration, types θ_1 and θ_2 are given up some surplus. This result is not surprising in that it is in line with the previous finding of GBS. The information rents are respectively written as:

$$
\begin{aligned}
R(\theta_1) = {}& \frac{p_1(\theta_1)}{p_1(\theta_2)} \left(C(q(\theta_3), \theta_3) - C(q(\theta_3), \theta_2) - L \frac{p_1(\theta_3) - p_1(\theta_2)}{p_1(\theta_3)} \right) \\
& + \frac{p_1(\theta_1)}{p_1(\theta_2)} \left(C(q(\theta_1), \theta_2) - C(q(\theta_1), \theta_1) - L \frac{p_1(\theta_2) - p_1(\theta_1)}{p_1(\theta_1)} \right)
\end{aligned}
\tag{41}
$$

and as:

$$
R(\theta_2) = C(q(\theta_3), \theta_3) - C(q(\theta_3), \theta_2) - L \frac{p_1(\theta_3) - p_1(\theta_2)}{p_1(\theta_3)}.
\tag{42}
$$

These expressions evidence that rents are conceded exactly because it would otherwise be impossible to satisfy (LIC) without violating (LL) ((33) is violated at the first-best allocation).

4.1.2 Condition (33) holds

The novel aspect to our second-best analysis is that when (33) holds, and hence satisfying the local incentive constraints is not an issue under limited liability, it is not necessary to decrease $\pi_n(\theta_2)$ to the minimum of $-L$ to retain all surplus from the agent. A simple way to see this is to check that both (41) and (42) are negative when (33) holds. Thus, by setting $\pi_n(\theta_2)$ strictly above $-L$, P can lessen the conflict between (37) and (38) without tightening (39) and (40), which eliminates the necessity to concede information rents. However, P does need an instrument to ensure that (37) and (38) hold at once. This instrument will be the quantity of type θ_2, which (37) and (38) depend upon. Specifically, P will need to adjust $q(\theta_2)$

so as to saturate (34), where $\theta^- = \theta_1$, $\theta = \theta_2$, $\theta^+ = \theta_3$. This requires lowering the difference:

$$\frac{C\left(q\left(\theta_2\right),\theta_2\right) - C\left(q\left(\theta_2\right),\theta_1\right)}{\frac{p_n(\theta_1)}{p_n(\theta_2)} - \frac{p_1(\theta_1)}{p_1(\theta_2)}} - \frac{C\left(q\left(\theta_2\right),\theta_3\right) - C\left(q\left(\theta_2\right),\theta_2\right)}{\frac{p_1(\theta_3)}{p_1(\theta_2)} - \frac{p_n(\theta_3)}{p_n(\theta_2)}}$$

below its first-best value. Hence, with $C_{q\theta}\left(q,\theta\right) > 0$, $q\left(\theta_2\right)$ will be distorted upwards at the second-best optimum.

4.2 A continuum of types

We complete our analysis coming back to the setting with a continuum of types. Take three types $\theta^- < \theta < \theta^+$ drawn from the feasible range Θ. Rewriting (37) and (38) with regards to these types and letting both θ^- and θ^+ tend to θ, (LIC) is reformulated as a first-degree differential equation (see Appendix H.2):

$$R'\left(\theta\right) = \frac{p_1'\left(\theta\right)}{p_1(\theta)}R\left(\theta\right) + \sum_{s\neq 1}\pi_s\left(\theta\right)p_s(\theta)\left(\frac{p_s'\left(\theta\right)}{p_s(\theta)} - \frac{p_1'\left(\theta\right)}{p_1\left(\theta\right)}\right) - C_\theta\left(q\left(\theta\right),\theta\right). \qquad (43)$$

The solution to the differential equation is given by the following expression of the rent:

$$R\left(\theta\right) = R\left(\overline{\theta}\right) + \int_\theta^{\overline{\theta}}\left[C_x\left(q\left(x\right),x\right) - \sum_{s\neq 1}\pi_s\left(x\right)p_s(x)\left(\frac{p_s'\left(x\right)}{p_s(x)} - \frac{p_1'\left(x\right)}{p_1(x)}\right)\right]\frac{p_1(\theta)}{p_1(x)}dx. \qquad (44)$$

Replacing (44) in the objective function of P, it becomes clear that, as long as no conflict arises between the incentive constraints whereby types θ^- and θ^+ are both unwilling to claim θ, the best is to set $R\left(\overline{\theta}\right) = 0$ together with $\pi_s\left(x\right) = -L$, $\forall s \neq 1$. Then, the information rent of type θ amounts to:

$$R\left(\theta\right) = \int_\theta^{\overline{\theta}}\frac{p_1(\theta)}{p_1(x)}\left(C_x\left(q\left(x\right),x\right) - L\frac{p_1'(x)}{p_1\left(x\right)}\right)dx, \ \forall\theta \neq \overline{\theta}, \qquad (45)$$

Further observe that $-\sum_{s=2}^n\pi_s\left(\theta^+\right)p_s\left(\theta^+\right)\left(\frac{p_1(\theta)}{p_1\left(\theta^+\right)} - \frac{p_s(\theta)}{p_s\left(\theta^+\right)}\right) > 0$ when $\pi_s\left(x\right) = -L \ \forall s \neq 1$. One can show that, when this condition holds, if cost exaggeration is prevented in a neighborhood of the true type, then it is prevented globally as well. Therefore, it suffices to focus on local incentive-compatibility. By insisting on the first-best allocation, P would be unable to satisfy (LIC) without violating (LL) because $L < C_x\left(q\left(x\right),x\right)\frac{p_1(x)}{p_1'(x)}$. She is thus forced to move away from first best and concede some surplus to the agent, calling for quantity distortions in turn. Once again, this result is identical to the finding of GBS, although here it is obtained in a different manner.

We are now left with exploring the case in which L is sufficiently high to allow for (LIC) to hold together with (LL) at the first-best quantity and, yet, first best is beyond reach because (34) is violated (recall that first-best implementation with our lottery requires that L be higher than with the lottery of GBS). We saw that the difficulty rests with the conflict between the incentive constraints whereby higher- and lower-order types are unwilling to claim θ. In light

of this, we rearrange the incentive constraints whereby types θ^- and θ^+ are unwilling the claim θ as:

$$\pi_n(\theta)\,p_n(\theta)$$

$$\leq \frac{R\left(\theta^-\right) - \frac{p_1\left(\theta^-\right)}{p_1(\theta)}R\left(\theta\right) - \left[C\left(q\left(\theta\right),\theta\right) - C\left(q\left(\theta\right),\theta^-\right)\right] - \sum_{s\neq 1,n}\pi_s\left(\theta\right)p_s(\theta)\left(\frac{p_s\left(\theta^-\right)}{p_s(\theta)} - \frac{p_1\left(\theta^-\right)}{p_1(\theta)}\right)}{\frac{p_n\left(\theta^-\right)}{p_n(\theta)} - \frac{p_1\left(\theta^-\right)}{p_1(\theta)}}$$

$$\pi_n(\theta)\,p_n(\theta)$$

$$\geq \frac{\frac{p_1\left(\theta^+\right)}{p_1(\theta)}R\left(\theta\right) - R\left(\theta^+\right) + \left[C\left(q\left(\theta\right),\theta\right) - C\left(q\left(\theta\right),\theta^+\right)\right] - \sum_{s\neq 1,n}\pi_s\left(\theta\right)p_s(\theta)\left(\frac{p_1\left(\theta^+\right)}{p_1(\theta)} - \frac{p_s\left(\theta^+\right)}{p_s(\theta)}\right)}{\frac{p_1\left(\theta^+\right)}{p_1(\theta)} - \frac{p_n\left(\theta^+\right)}{p_n(\theta)}}.$$

Extracting the following expression of $\pi_n(\theta)\,p_n(\theta)$ from the local incentive constraint:

$$\pi_n(\theta)\,p_n(\theta) = \frac{\frac{p_1'(\theta)}{p_1(\theta)}R(\theta) - R'(\theta) - C_\theta\left(q\left(\theta\right),\theta\right) - \sum_{s\neq 1,n}\pi_s\left(\theta\right)p_s(\theta)\left(\frac{p_1'(\theta)}{p_1(\theta)} - \frac{p_s'(\theta)}{p_s(\theta)}\right)}{\frac{p_1'(\theta)}{p_1(\theta)} - \frac{p_n'(\theta)}{p_n(\theta)}},$$

we see that the incentive constraints of types θ^- and θ^+ are jointly satisfied only if:

$$\frac{\frac{p_1\left(\theta^+\right)}{p_1(\theta)}R\left(\theta\right) - R\left(\theta^+\right)}{\frac{p_1\left(\theta^+\right)}{p_1(\theta)} - \frac{p_n\left(\theta^+\right)}{p_n(\theta)}} - \frac{R\left(\theta^-\right) - \frac{p_1\left(\theta^-\right)}{p_1(\theta)}R\left(\theta\right)}{\frac{p_n\left(\theta^-\right)}{p_n(\theta)} - \frac{p_1\left(\theta^-\right)}{p_1(\theta)}}$$

$$\leq L\sum_{s\neq 1,n}p_s(\theta)\left(\frac{\frac{p_s\left(\theta^-\right)}{p_s(\theta)} - \frac{p_1\left(\theta^-\right)}{p_1(\theta)}}{\frac{p_n\left(\theta^-\right)}{p_n(\theta)} - \frac{p_1\left(\theta^-\right)}{p_1(\theta)}} - \frac{\frac{p_1\left(\theta^+\right)}{p_1(\theta)} - \frac{p_s\left(\theta^+\right)}{p_s(\theta)}}{\frac{p_1\left(\theta^+\right)}{p_1(\theta)} - \frac{p_n\left(\theta^+\right)}{p_n(\theta)}}\right)$$

$$- \left(\frac{C\left(q\left(\theta\right),\theta\right) - C\left(q\left(\theta\right),\theta^-\right)}{\frac{p_n\left(\theta^-\right)}{p_n(\theta)} - \frac{p_1\left(\theta^-\right)}{p_1(\theta)}} - \frac{C\left(q\left(\theta\right),\theta^+\right) - C\left(q\left(\theta\right),\theta\right)}{\frac{p_1\left(\theta^+\right)}{p_1(\theta)} - \frac{p_n\left(\theta^+\right)}{p_n(\theta)}}\right).$$

This condition generalizes (27) to the case in which the information rents may be positive. As in the three-type setting, this condition is binding at optimum for any triplet $\left\{\theta^-,\theta,\theta^+\right\}$. To satisfy it, P does not need to concede an information rent to the agent. The optimal strategy is to set $R\left(\theta^-\right) = R\left(\theta\right) = R\left(\theta^+\right) = 0$ and distort output levels upwards for all but the extreme types.

Proposition 2 *If (33) is violated, then the second-best lottery is such that $\pi_s(\theta) = -L$, $\forall\theta$, $\forall s \neq 1$. The agent is conceded an information rent given by (45) and the output level $q(\theta)$ is distorted downwards $\forall\theta \neq \underline{\theta}$. If (33) is satisfied whereas (34) is violated, then the second-best lottery is such that $\pi_s(\theta) = -L$, $\forall\theta$, $\forall s \neq 1, n$ and $\pi_n(\theta) > -L$, $\forall\theta$. No information rent is conceded to the agent and the output level $q(\theta)$ is distorted upwards $\forall\theta \neq \underline{\theta}, \overline{\theta}$.*

This result naturally extends the second-best result of GBS to the case in which (32) is violated. The novel finding that no information rent is conceded when (32) is violated is perhaps

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less intuitive. It emphasizes the fact that different results are possible when the departure from first best is due to the presence of limited liability on the agent's side rather than to violations of the full-rank condition as considered by Bose and Zhao [1]. These authors assume that the vector of conditional probabilities of some intermediate type is a linear combination of those of the other types. Hence, the full-rank condition is violated and first best might not be at reach. The intermediate type will then be extracted all surplus at optimum and at most one of the extreme types will receive a rent. However, in that setting, restrictions on the lottery to be adopted are due to the linear combination of probability vectors, whereas there is no bound to the size of the profits. When restrictions follow from limited liability, the agent is assigned an information rent only if the losses he can be inflicted are so low that (33) is violated.

5   Conclusion

In a principal-agent model with correlated information and limited liability on the agent's side, we showed that focusing on the full-rank condition, the most common approach in the literature, is not necessarily the best approach. Provided that there exist at least three informational signals, the conditional probabilities of two of them displaying a monotonicity property, it is enough to verify that the liability of the agent is sufficiently high to ascertain whether or not first best is implementable, which is very useful in applications. Whereas Bose and Zhao [1] investigate first-best implementation when the full-rank condition does not hold, we proved that the possibility of attaining the first-best outcome under limited liability is not necessarily determined by the way in which the conditional probabilities of the signals depart from the full-rank condition. Moreover, the existence of an exact relationship between the extent of the liability and the admissible degree of concavity of the cost function (when this is not convex in type) involves that the set of technologies for which first best is at reach under limited liability is richer than that considered by GBS. Noticeably, these results would carry over if limited liability were a constraint on the transfer payments to the agent rather than on his profits, a possibility considered by Demougin and Garvie [3]. The reason is that in either case limited liability imposes restrictions on the total transfer from the principal to the agent, and not on how that transfer is structured in terms of fixed and variable payment.[13]

As a general view, our study contributes to shedding light on how to attain incentive-compatibility in situations in which the principal faces more than two possible types of agent and there are more than two informational signals to be used in contractual design. Our findings point to the conclusion that it might be with loss of generality to restrict attention to the two-type case, or to a binary signal, when exploring principal-agent relationships with correlated information and limited liability.

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[13]Note however that results could be qualitatively different if limited liability were in the form of a bound to the fixed and the variable transfer separately rather than to the overall compensation.

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A   Full-rank condition and assumption (1)

Suppose that the vector $\mathbf{p}(\theta_1)$ lies in the convex hull generated by the other probability vectors. Then, there exists a vector $(\lambda_2, ..., \lambda_T)$, where $\lambda_t \in [0,1]$ $\forall t \in \{2, ..., T\}$ and $\sum_{t=2}^{T} \lambda_t = 1$, such that:

$$p_s(\theta_1) = \lambda_2 p_s(\theta_2) + ... + \lambda_T p_s(\theta_T), \ \forall s \in N.$$

Let us use this for $s = 1$ and $s \neq 1$ together with (1). We get:

$$
\begin{aligned}
p_1(\theta_1) &= \lambda_2 p_1(\theta_2) + ... + \lambda_T p_1(\theta_T) \\
&\Leftrightarrow \frac{p_1(\theta_1)}{p_1(\theta_2)} = \lambda_2 \frac{p_1(\theta_2)}{p_1(\theta_2)} + ... + \lambda_T \frac{p_1(\theta_T)}{p_1(\theta_2)} > \lambda_2 \frac{p_s(\theta_2)}{p_s(\theta_2)} + ... + \lambda_T \frac{p_s(\theta_T)}{p_s(\theta_2)} = \frac{p_s(\theta_1)}{p_s(\theta_2)}.
\end{aligned}
$$

The inequality $\frac{p_1(\theta_1)}{p_1(\theta_2)} > \frac{p_s(\theta_1)}{p_s(\theta_2)}$ contradicts (1). Similarly, suppose that there exists a vector $(\lambda_1, ..., \lambda_{T-1})$, where $\lambda_t \in [0,1]$ $\forall t \in \{1, ..., T-1\}$ and $\sum_{t=1}^{T-1} \lambda_t = 1$, such that:

$$p_s(\theta_T) = \lambda_1 p_s(\theta_1) + ... + \lambda_{T-1} p_s(\theta_{T-1}), \ \forall s \in N.$$

Let us use this for $s = 1$ and $s \neq 1$ together with (1). We get:

$$p_1(\theta_T) = \lambda_1 p_1(\theta_1) + ... + \lambda_{T-1} p_1(\theta_{T-1})$$
$$\Leftrightarrow$$
$$\frac{p_1(\theta_T)}{p_1(\theta_{T-1})} = \lambda_1 \frac{p_1(\theta_1)}{p_1(\theta_{T-1})} + ... + \lambda_{T-1} \frac{p_1(\theta_{T-1})}{p_1(\theta_{T-1})} < \lambda_1 \frac{p_s(\theta_1)}{p_s(\theta_{T-1})} + ... + \lambda_{T-1} \frac{p_s(\theta_{T-1})}{p_s(\theta_{T-1})} = \frac{p_s(\theta_T)}{p_s(\theta_{T-1})}.$$

The inequality $\frac{p_1(\theta_T)}{p_1(\theta_{T-1})} < \frac{p_s(\theta_T)}{p_s(\theta_{T-1})}$ contradicts (1).

Next take the vector $\mathbf{p}(\theta_t)$, where $t \notin \{1, T\}$, to lie in the convex hull generated by the probability vectors of the other types. This is equivalent to telling that there exists a vector $(\lambda_1, ..., \lambda_{t-1}, \lambda_{t+1}, ..., \lambda_T)$, where $\lambda_t \in [0, 1] \ \forall t \in \{1, ..., t-1, t+1, ..., T\}$ and $\sum_{t' \neq t} \lambda_{t'} = 1$, such that:

$$p_s(\theta_t) = \lambda_1 p_s(\theta_1) + ... + \lambda_{t-1} p_s(\theta_{t-1}) + \lambda_{t+1} p_s(\theta_{t+1}) + ... + \lambda_T p_s(\theta_T)$$
$$\Leftrightarrow$$
$$\frac{p_s(\theta_t)}{p_s(\theta_{t+1})} = \lambda_1 \frac{p_s(\theta_1)}{p_s(\theta_{t+1})} + ... + \lambda_{t-1} \frac{p_s(\theta_{t-1})}{p_s(\theta_{t+1})} + \lambda_{t+1} \frac{p_s(\theta_{t+1})}{p_s(\theta_{t+1})} + ... + \lambda_T \frac{p_s(\theta_T)}{p_s(\theta_{t+1})}. \quad (46)$$

By taking $\mathbf{p}(\theta_t)$ such that

$$\frac{p_{s'}(\theta_t)}{p_{s'}(\theta_{t+1})}$$
$$> \frac{p_1(\theta_t)}{p_1(\theta_{t+1})}$$
$$> \frac{p_{s'}(\theta_t)}{p_{s'}(\theta_{t+1})} + \lambda_1 \left(\frac{p_1(\theta_1)}{p_1(\theta_{t+1})} - \frac{p_{s'}(\theta_1)}{p_{s'}(\theta_{t+1})} \right) + ... + \lambda_{t-1} \left(\frac{p_1(\theta_{t-1})}{p_1(\theta_{t+1})} - \frac{p_{s'}(\theta_{t-1})}{p_{s'}(\theta_{t+1})} \right), \ \forall s' \neq 1,$$

both (1) and (46) are satisfied. To see this, first use (46) for $s = 1$ to rewrite the second inequality here above as:

$$\lambda_{t+1} \frac{p_1(\theta_{t+1})}{p_1(\theta_{t+1})} + ... + \lambda_T \frac{p_1(\theta_T)}{p_1(\theta_{t+1})} > \frac{p_{s'}(\theta_t)}{p_{s'}(\theta_{t+1})} + \lambda_1 \left(-\frac{p_{s'}(\theta_1)}{p_{s'}(\theta_{t+1})} \right) + ... + \lambda_{t-1} \left(-\frac{p_{s'}(\theta_{t-1})}{p_{s'}(\theta_{t+1})} \right)$$
$$= \lambda_{t+1} \frac{p_{s'}(\theta_{t+1})}{p_{s'}(\theta_{t+1})} + ... + \lambda_T \frac{p_{s'}(\theta_T)}{p_{s'}(\theta_{t+1})}$$

Then use (46) for s' to rewrite:

$$\lambda_{t+1} \frac{p_1(\theta_{t+1})}{p_1(\theta_{t+1})} + ... + \lambda_T \frac{p_1(\theta_T)}{p_1(\theta_{t+1})} > \lambda_{t+1} \frac{p_{s'}(\theta_{t+1})}{p_{s'}(\theta_{t+1})} + ... + \lambda_T \frac{p_{s'}(\theta_T)}{p_{s'}(\theta_{t+1})},$$

which is true by assumption (1).

B   Two and three signals

We first identify (GIC) for the general case with n signals and then specify it for 2 and 3 signals.

B.1 (GIC) with n signals

Using $\widetilde{\pi}_s \left(\theta \,|\theta'\right) = t_s \left(\theta\right) - C \left(q \left(\theta\right), \theta'\right)$ and $\pi_s \left(\theta\right) = \widetilde{\pi}_s \left(\theta \,|\theta\right)$, we have:

$$\mathbb{E}_s \left[\widetilde{\pi}_s \left(\theta \,|\theta'\right)\right] = \sum_{s=1}^{n} \pi_s \left(\theta\right) p_s \left(\theta'\right) + C \left(q \left(\theta\right), \theta\right) - C \left(q \left(\theta\right), \theta'\right).$$

Because full surplus extraction requires $\sum_{s=1}^{n} \pi_s \left(\theta\right) p_s \left(\theta\right) = 0$, this is rewritten as (GIC). Further using $\sum_{s=1}^{n} \pi_s \left(\theta\right) p_s \left(\theta\right) = 0 \Leftrightarrow \pi_1 \left(\theta\right) = -\sum_{s=2}^{n} \pi_s \left(\theta\right) \frac{p_s(\theta)}{p_1(\theta)}$, (GIC) is further rewritten as:

$$C \left(q \left(\theta\right), \theta\right) - C \left(q \left(\theta\right), \theta'\right) \le \sum_{s \neq 1, n} \pi_s \left(\theta\right) p_s \left(\theta\right) \left(\frac{p_1 \left(\theta'\right)}{p_1(\theta)} - \frac{p_s \left(\theta'\right)}{p_s(\theta)}\right) + \pi_n \left(\theta\right) p_n(\theta) \left(\frac{p_1 \left(\theta'\right)}{p_1(\theta)} - \frac{p_n \left(\theta'\right)}{p_n(\theta)}\right),$$

hence:

$$\pi_n \left(\theta\right) p_n(\theta) \left(\frac{p_1 \left(\theta'\right)}{p_1(\theta)} - \frac{p_n \left(\theta'\right)}{p_n(\theta)}\right) \ge C \left(q \left(\theta\right), \theta\right) - C \left(q \left(\theta\right), \theta'\right) - \sum_{s \neq 1, n} \pi_s \left(\theta\right) p_s \left(\theta\right) \left(\frac{p_1 \left(\theta'\right)}{p_1(\theta)} - \frac{p_s \left(\theta'\right)}{p_s(\theta)}\right).$$
$$(47)$$

Recall that, by assumption, $\frac{p_1(\theta')}{p_1(\theta)} > \frac{p_n(\theta')}{p_n(\theta)}$ if and only if $\theta' > \theta$. Using this equivalence for $\theta^- < \theta$ and $\theta^+ > \theta$, (47) is respectively rewritten as:

$$\pi_n \left(\theta\right) p_n(\theta) \le -\frac{C \left(q \left(\theta\right), \theta\right) - C \left(q \left(\theta\right), \theta^-\right)}{\frac{p_n(\theta^-)}{p_n(\theta)} - \frac{p_1(\theta^-)}{p_1(\theta)}} - \sum_{s \neq 1, n} \pi_s \left(\theta\right) p_s \left(\theta\right) \frac{\frac{p_s(\theta^-)}{p_s(\theta)} - \frac{p_1(\theta^-)}{p_1(\theta)}}{\frac{p_n(\theta^-)}{p_n(\theta)} - \frac{p_1(\theta^-)}{p_1(\theta)}} \qquad (48)$$

and

$$\pi_n \left(\theta\right) p_n(\theta) \ge -\frac{C \left(q \left(\theta\right), \theta^+\right) - C \left(q \left(\theta\right), \theta\right)}{\frac{p_1(\theta^+)}{p_1(\theta)} - \frac{p_n(\theta^+)}{p_n(\theta)}} - \sum_{s \neq 1, n} \pi_s \left(\theta\right) p_s \left(\theta\right) \frac{\frac{p_1(\theta^+)}{p_1(\theta)} - \frac{p_s(\theta^+)}{p_s(\theta)}}{\frac{p_1(\theta^+)}{p_1(\theta)} - \frac{p_n(\theta^+)}{p_n(\theta)}}. \qquad (49)$$

B.2 Two signals

When $n = 2$, (48) and (49) specify as (13), (15) and (14).

To check that the global incentive constraints are satisfied, we need to verify that (13) and (15) are respectively satisfied for $\theta_t = \theta_3$ and $\theta_t = \theta_1$, if they are for θ_2. This is the case when:

$$\frac{C \left(q \left(\theta_3\right), \theta_3\right) - C \left(q \left(\theta_3\right), \theta_1\right)}{C \left(q \left(\theta_3\right), \theta_3\right) - C \left(q \left(\theta_3\right), \theta_2\right)} \le \frac{\frac{p_2(\theta_1)}{p_2(\theta_3)} - \frac{p_1(\theta_1)}{p_1(\theta_3)}}{\frac{p_2(\theta_2)}{p_2(\theta_3)} - \frac{p_1(\theta_2)}{p_1(\theta_3)}} \qquad (50)$$

$$\frac{C \left(q \left(\theta_1\right), \theta_2\right) - C \left(q \left(\theta_1\right), \theta_1\right)}{C \left(q \left(\theta_1\right), \theta_3\right) - C \left(q \left(\theta_1\right), \theta_1\right)} \le \frac{\frac{p_1(\theta_2)}{p_1(\theta_1)} - \frac{p_2(\theta_2)}{p_2(\theta_1)}}{\frac{p_1(\theta_3)}{p_1(\theta_1)} - \frac{p_2(\theta_3)}{p_2(\theta_1)}}. \qquad (51)$$

Using $p_2(\cdot) = 1 - p_1(\cdot)$, these conditions are rewritten as:

$$\frac{p_1(\theta_3) - p_1(\theta_1)}{p_1(\theta_3) - p_1(\theta_2)} \geq \frac{C(q(\theta_3), \theta_3) - C(q(\theta_3), \theta_1)}{C(q(\theta_3), \theta_3) - C(q(\theta_3), \theta_2)}$$

$$\frac{p_1(\theta_2) - p_1(\theta_1)}{p_1(\theta_3) - p_1(\theta_1)} \geq \frac{C(q(\theta_1), \theta_2) - C(q(\theta_1), \theta_1)}{C(q(\theta_1), \theta_3) - C(q(\theta_1), \theta_1)}.$$

Replacing $p_1(\theta_3) - p_1(\theta_1)$ with $p_1(\theta_3) - p_1(\theta_2) + p_1(\theta_2) - p_1(\theta_1)$ and $C(q(\theta_3), \theta_3) - C(q(\theta_3), \theta_1)$ with $C(q(\theta_3), \theta_3) - C(q(\theta c_3), \theta_2) + C(q(\theta_3), \theta_2) - C(q(\theta_3), \theta_1)$, the two conditions further become:

$$\frac{C(q(\theta_3), \theta_2) - C(q(\theta_3), \theta_1)}{C(q(\theta_3), \theta_3) - C(q(\theta_3), \theta_2)} \leq \frac{p_1(\theta_2) - p_1(\theta_1)}{p_1(\theta_3) - p_1(\theta_2)} \tag{52}$$

$$\frac{C(q(\theta_1), \theta_2) - C(q(\theta_1), \theta_1)}{C(q(\theta_1), \theta_3) - C(q(\theta_1), \theta_2)} \leq \frac{p_1(\theta_2) - p_1(\theta_1)}{p_1(\theta_3) - p_1(\theta_2)}, \tag{53}$$

which are equivalent to (7) for specified quantities $q(\cdot)$.

B.3 Three signals

Specify (48) and (49) for $n = 3$, $\theta = \theta_2$, and, respectively, $\theta^- = \theta_1$ and $\theta^+ = \theta_3$. $\exists \pi_3(\theta_2)$ that satisfies both (48) and (49) if and only if (21) is satisfied. Resting on the equality:

$$\frac{p_i(\theta') p_1(\theta) - p_1(\theta') p_i(\theta)}{p_i(\theta) p_1(\theta)} = \frac{p_1(\theta) - p_1(\theta')}{p_1(\theta)} - \frac{p_i(\theta) - p_i(\theta')}{p_i(\theta)},$$

the term multiplied by $\pi_2(\theta_2)$ in (21) is negative if and only if (22) holds.

C Derivation of (LIC) and (23)

Recall $\widetilde{\pi}_s(\theta \,|\, \theta') = t_s(\theta) - C(q(\theta), \theta')$ and

$$\mathbb{E}_s[\widetilde{\pi}_s(\theta \,|\, \theta')] \equiv \sum_{s=1}^{n} (t_s(\theta) - C(q(\theta), \theta')) p_s(\theta'). \tag{54}$$

The first-order condition of the agent's problem, evaluated at $\theta' = \theta$, is given by:

$$\sum_{s=1}^{n} (t'_s(\theta) - C_q(q(\theta), \theta) q_\theta(\theta)) p_s(\theta) = 0. \tag{55}$$

From $t_s(\theta) = \pi_s(\theta) + C(q(\theta), \theta)$, we compute $t'_s(\theta) = \pi'_s(\theta) + C_q(q(\theta), \theta) q_\theta(\theta) + C_\theta(q(\theta), \theta)$, which we then replace into (55) to get:

$$C_\theta(q(\theta), \theta) = -\sum_{s=1}^{n} \pi'_s(\theta) p_s(\theta). \tag{56}$$

Because $\sum_{s=1}^{n} \pi_s(\theta) p_s(\theta) = 0$, $\forall \theta$, implies $-\sum_{s=1}^{n} \pi'_s(\theta) p_s(\theta) = \sum_{s=1}^{n} \pi_s(\theta) p'_s(\theta)$, $\forall \theta$, (56) is further rewritten as (LIC).

Rewrite (48) and (49) as:

$$\pi_n(\theta) p_n(\theta) \leq -\frac{\frac{C(q(\theta),\theta) - C\left(q(\theta),\theta^-\right)}{\theta-\theta^-}}{\frac{p_1(\theta)-p_1(\theta^-)}{\theta-\theta^-}}{p_1(\theta)} - \frac{\frac{p_n(\theta)-p_n(\theta^-)}{\theta-\theta^-}}{p_n(\theta)}} - \sum_{s \neq 1,n} \pi_s(\theta) p_s(\theta) \frac{\frac{\frac{p_1(\theta)-p_1(\theta^-)}{\theta-\theta^-}}{p_1(\theta)} - \frac{\frac{p_s(\theta)-p_s(\theta^-)}{\theta-\theta^-}}{p_s(\theta)}}{\frac{\frac{p_1(\theta)-p_1(\theta^-)}{\theta-\theta^-}}{p_1(\theta)} - \frac{\frac{p_n(\theta)-p_n(\theta^-)}{\theta-\theta^-}}{p_n(\theta)}}$$

and

$$\pi_n(\theta) p_n(\theta) \geq -\frac{\frac{C\left(q(\theta),\theta^+\right) - C(q(\theta),\theta)}{\theta^+-\theta}}{\frac{p_1(\theta^+)-p_1(\theta)}{\theta^+-\theta}}{p_1(\theta)} - \frac{\frac{p_n(\theta^+)-p_n(\theta)}{\theta^+-\theta}}{p_n(\theta)}} - \sum_{s \neq 1,n} \pi_s(\theta) p_s(\theta) \frac{\frac{\frac{p_1(\theta^+)-p_1(\theta)}{\theta^+-\theta}}{p_1(\theta)} - \frac{\frac{p_s(\theta^+)-p_s(\theta)}{\theta^+-\theta}}{p_s(\theta)}}{\frac{\frac{p_1(\theta^+)-p_1(\theta)}{\theta^+-\theta}}{p_1(\theta)} - \frac{\frac{p_n(\theta^+)-p_n(\theta)}{\theta^+-\theta}}{p_n(\theta)}}.$$

Taking the limit for $\theta^- \to \theta$ and for $\theta^+ \to \theta$, the conditions are both satisfied if and only if $\pi_n(\theta)$ satisfies (23).

D Proof of Lemma 1

Suppose that some profit $\pi_i(\theta)$ is changed by ε. Accordingly, $\pi_j(\theta)$ is changed by ζ and $\pi_k(\theta)$ by δ such that (PC) is still saturated and the right-hand side of (LIC) does not vary. Dropping the argument θ everywhere for the sake of shortness, this requires:

$$\zeta p_j = -\varepsilon p_i - \delta p_k \Leftrightarrow \zeta = -\varepsilon \frac{p_i}{p_j} - \delta \frac{p_k}{p_j}$$

$$\delta p'_k = -\zeta p'_j - \varepsilon p'_i \Leftrightarrow \delta = -\zeta \frac{p'_j}{p'_k} - \varepsilon \frac{p'_i}{p'_k}.$$

Replacing the expression of δ in that of ζ, we obtain:

$$\zeta = -\varepsilon \frac{p_i}{p_j} \frac{\frac{p'_i}{p_i} - \frac{p'_k}{p_k}}{\frac{p'_j}{p_j} - \frac{p'_k}{p_k}}. \tag{57}$$

Replacing (57) in the expression of δ, we further obtain:

$$\delta = \varepsilon \frac{p_i}{p_k} \frac{\frac{p'_i}{p_i} - \frac{p'_j}{p_j}}{\frac{p'_j}{p_j} - \frac{p'_k}{p_k}}. \tag{58}$$

Using (24) in (57) and (58), we deduce that $\operatorname{Si} gn(\zeta) \neq \operatorname{Si} gn(\delta)$.

E Proof of Lemma 2

Taking the expression of $\pi_n(\theta) p_n(\theta)$ from (23), pugging into (48) and making use of the inequalities $\frac{p'_1(\theta)}{p_1(\theta)} > \frac{p'_n(\theta)}{p_n(\theta)}$ and $\frac{p_n(\theta^-)}{p_n(\theta)} > \frac{p_1(\theta^-)}{p_1(\theta)}$ to rearrange, (48) is rewritten as (25). Similarly,

(49) is rewritten as (26).

F   Proof of Lemma 3

The necessity of (27) is obvious. To show sufficiency, we first let θ^+ tend to θ. Applying de L'Hopital's rule yields:

$$\frac{\frac{p_1(\theta^+)}{p_1(\theta)} - \frac{p_s(\theta^+)}{p_s(\theta)}}{\frac{p_1(\theta^+)}{p_1(\theta)} - \frac{p_n(\theta^+)}{p_n(\theta)}} = \frac{\frac{p_1'(\theta)}{p_1(\theta)} - \frac{p_s'(\theta)}{p_s(\theta)}}{\frac{p_1'(\theta)}{p_1(\theta)} - \frac{p_n'(\theta)}{p_n(\theta)}}.$$

Using this in (27), we obtain (25). Similarly, as θ^- tends to θ :

$$\frac{\frac{p_1(\theta^-)}{p_1(\theta)} - \frac{p_s(\theta^-)}{p_s(\theta)}}{\frac{p_1(\theta^-)}{p_1(\theta)} - \frac{p_n(\theta^-)}{p_n(\theta)}} = \frac{\frac{p_1'(\theta)}{p_1(\theta)} - \frac{p_s'(\theta)}{p_s(\theta)}}{\frac{p_1'(\theta)}{p_1(\theta)} - \frac{p_n'(\theta)}{p_n(\theta)}}.$$

Using this in (27), we obtain (26). Hence, (27) is sufficient as well.

G   Proof of Proposition 1 and corollaries in the first-best setting

G.1   Proof of Proposition 1

Derivation of (34)

We see that

$$\frac{d}{d\theta^+}\left(\frac{\frac{p_1(\theta^+)}{p_1(\theta)} - \frac{p_s(\theta^+)}{p_s(\theta)}}{\frac{p_1(\theta^+)}{p_1(\theta)} - \frac{p_n(\theta^+)}{p_n(\theta)}}\right) < 0$$

if and only if

$$\frac{\frac{p_1'(\theta^+)}{p_1(\theta)} - \frac{p_s'(\theta^+)}{p_s(\theta)}}{\frac{p_1(\theta^+)}{p_1(\theta)} - \frac{p_s(\theta^+)}{p_s(\theta)}} < \frac{\frac{p_1'(\theta^+)}{p_1(\theta)} - \frac{p_n'(\theta^+)}{p_n(\theta)}}{\frac{p_1(\theta^+)}{p_1(\theta)} - \frac{p_n(\theta^+)}{p_n(\theta)}}. \tag{59}$$

Multiplying the numerator by $\theta^+ - \theta$ in both sides, subtracting 1 from each side and manipulating further, (59) becomes:

$$\frac{\frac{p_s(\theta^+) - p_s'(\theta^+)(\theta^+ - \theta)}{p_s(\theta)} - \frac{p_1(\theta^+) - p_1'(\theta^+)(\theta^+ - \theta)}{p_1(\theta)}}{\frac{p_1(\theta^+)}{p_1(\theta)} - \frac{p_s(\theta^+)}{p_s(\theta)}} < \frac{\frac{p_n(\theta^+) - p_n'(\theta^+)(\theta^+ - \theta)}{p_n(\theta)} - \frac{p_1(\theta^+) - p_1'(\theta^+)(\theta^+ - \theta)}{p_1(\theta)}}{\frac{p_1(\theta^+)}{p_1(\theta)} - \frac{p_n(\theta^+)}{p_n(\theta)}}.$$

Using the definition of $\rho_s(\theta', \theta)$, this is rewritten as:

$$\frac{\rho_s(\theta^+, \theta) - \rho_1(\theta^+, \theta)}{\frac{p_1(\theta^+)}{p_1(\theta)} - \frac{p_s(\theta^+)}{p_s(\theta)}} < \frac{\rho_n(\theta^+, \theta) - \rho_1(\theta^+, \theta)}{\frac{p_1(\theta^+)}{p_1(\theta)} - \frac{p_n(\theta^+)}{p_n(\theta)}}, \tag{60}$$

which is satisfied by assumption.

We also see that:

$$\frac{d}{d\theta^-}\left(\frac{\frac{p_1(\theta^-)}{p_1(\theta)} - \frac{p_s(\theta^-)}{p_s(\theta)}}{\frac{p_1(\theta^-)}{p_1(\theta)} - \frac{p_n(\theta^-)}{p_n(\theta)}}\right) < 0$$

if and only if

$$\frac{\frac{p_1'(\theta^-)}{p_1(\theta)} - \frac{p_s'(\theta^-)}{p_s(\theta)}}{\frac{p_s(\theta^-)}{p_s(\theta)} - \frac{p_1(\theta^-)}{p_1(\theta)}} > \frac{\frac{p_1'(\theta^-)}{p_1(\theta)} - \frac{p_n'(\theta^-)}{p_n(\theta)}}{\frac{p_n(\theta^-)}{p_n(\theta)} - \frac{p_1(\theta^-)}{p_1(\theta)}}. \tag{61}$$

Multiply both sides by $(\theta - \theta^-)$, subtract from either side and rearrange to obtain:

$$\frac{\frac{p_1(\theta^-)+p_1'(\theta^-)(\theta-\theta^-)}{p_1(\theta)} - \frac{p_s(\theta^-)+p_s'(\theta^-)(\theta-\theta^-)}{p_s(\theta)}}{\frac{p_s(\theta^-)}{p_s(\theta)} - \frac{p_1(\theta^-)}{p_1(\theta)}} > \frac{\frac{p_1(\theta^-)+p_1'(\theta^-)(\theta-\theta^-)}{p_1(\theta)} - \frac{p_n(\theta^-)+p_n'(\theta^-)(\theta-\theta^-)}{p_n(\theta)}}{\frac{p_n(\theta^-)}{p_n(\theta)} - \frac{p_1(\theta^-)}{p_1(\theta)}}.$$

Resting on the definition of ρ, this is rewritten as:

$$\frac{\rho_s(\theta^-,\theta) - \rho_1(\theta^-,\theta)}{\frac{p_s(\theta^-)}{p_s(\theta)} - \frac{p_1(\theta^-)}{p_1(\theta)}} < \frac{\rho_n(\theta^-,\theta) - \rho_1(\theta^-,\theta)}{\frac{p_n(\theta^-)}{p_n(\theta)} - \frac{p_1(\theta^-)}{p_1(\theta)}}, \tag{62}$$

which is satisfied by assumption.

Therefore, we have:

$$\frac{d}{d\theta^+}\left(\frac{\frac{p_1(\theta^+)}{p_1(\theta)} - \frac{p_s(\theta^+)}{p_s(\theta)}}{\frac{p_1(\theta^+)}{p_1(\theta)} - \frac{p_n(\theta^+)}{p_n(\theta)}}\right) < 0 \text{ together with } \frac{d}{d\theta^-}\left(\frac{\frac{p_1(\theta^-)}{p_1(\theta)} - \frac{p_s(\theta^-)}{p_s(\theta)}}{\frac{p_1(\theta^-)}{p_1(\theta)} - \frac{p_n(\theta^-)}{p_n(\theta)}}\right) < 0,$$

involving that the difference

$$\frac{\frac{p_1(\theta^+)}{p_1(\theta)} - \frac{p_s(\theta^+)}{p_s(\theta)}}{\frac{p_1(\theta^+)}{p_1(\theta)} - \frac{p_n(\theta^+)}{p_n(\theta)}} - \frac{\frac{p_1(\theta^-)}{p_1(\theta)} - \frac{p_s(\theta^-)}{p_s(\theta)}}{\frac{p_1(\theta^-)}{p_1(\theta)} - \frac{p_n(\theta^-)}{p_n(\theta)}}$$

is greatest as θ^- tends to θ and θ^+ tends to θ. For such values of θ^- and θ^+, the difference here above is found to be zero (by applying de L'Hopital's rule). Hence, for all pairs of types, the difference is non-positive. In definitive, for any given pair $\{\theta^-, \theta^+\}$ such that $\theta^- < \theta < \theta^+$, (27) is weakest if $\pi_s(\theta) = -L, \forall s \neq 1, n$. Substituting this value in (27) and rearranging yields (34).

Proof of (32) and (33)

Setting $\pi_s(\theta) = \pi_n(\theta)$ in (48), we see that $\pi_n(\theta) \geq -L$ if and only if (33) is satisfied. The fact that no other lottery satisfies (LL), if (LL) is not satisfied by $\boldsymbol{\pi}^1(\theta)$ (the lottery such that $\pi_s(\theta)$ is equal $\forall s \neq 1$), follows from Lemma 1.

Setting $\pi_s(\theta) = \pi_n(\theta)$ in (23) and then plugging the resulting expression of $\pi_n(\theta)$, we see that (48) and (49) are jointly satisfied if and only if so is (32).

G.2 Proof of Corollary 1

Using $\pi_s(\theta) = -L$ in (23), $\pi_n(\theta)$ is rewritten as:

$$\pi_n(\theta) = -\frac{C_\theta(q(\theta),\theta) - L\sum_{s\neq 1,n} p_s(\theta)\left(\frac{p_1'(\theta)}{p_1(\theta)} - \frac{p_i'(\theta)}{p_i(\theta)}\right)}{p_n(\theta)\left(\frac{p_1'(\theta)}{p_1(\theta)} - \frac{p_n'(\theta)}{p_n(\theta)}\right)}.$$

Replacing $\sum_{s\neq 1,n} p_i(\theta) = 1 - p_1(\theta) - p_n(\theta)$ and $\sum_{s\neq 1,n} p_s'(\theta) = -p_1'(\theta) - p_n'(\theta)$, $\pi_n(\theta)$ is further rewritten as (30).

Recalling that $\pi_1(\theta) = -\sum_{s=2}^n \pi_s(\theta)\frac{p_i(\theta)}{p_1(\theta)}$ because $\sum_{s=1}^n \pi_i(\theta)p_i(\theta) = 0$, and using $\pi_s(\theta) = -L$ and (30) in the expression of $\pi_1(\theta)$ we find:

$$
\begin{aligned}
\pi_1(\theta) &= -\sum_{s\neq 1,n}^n \pi_s(\theta)\frac{p_i(\theta)}{p_1(\theta)} - \pi_n(\theta)\frac{p_n(\theta)}{p_1(\theta)}\\[2mm]
&= \frac{L}{p_1(\theta)}\sum_{s\neq 1,n}^n p_i(\theta) - \left(\frac{L\frac{p_1'(\theta)}{p_1(\theta)} - C_\theta(q(\theta),\theta)}{p_n(\theta)\left(\frac{p_1'(\theta)}{p_1(\theta)} - \frac{p_n'(\theta)}{p_n(\theta)}\right)} - L\right)\frac{p_n(\theta)}{p_1(\theta)}
\end{aligned}
$$

Replacing again $\sum_{s\neq 1,n} p_i(\theta) = 1 - p_1(\theta) - p_n(\theta)$, $\pi_1(\theta)$ is further rewritten as (29).

We are left with checking that $\pi_1(\theta) \geq -L$ and $\pi_n(\theta) \geq -L$. The former is true because $p_n'(\theta) < 0$. The latter is implied by $\frac{p_1'(\theta)}{p_1(\theta)} > \frac{p_n'(\theta)}{p_n(\theta)}$ together with $C_\theta(q(\theta),\theta)\frac{p_1(\theta)}{p_1'(\theta)} \leq L$, which is implied by (33).

G.3 Proof of Corollary 2

Recall that by applying de L'Hopital's rule one has:

$$\lim_{\theta^+\to\theta} \frac{\frac{p_1(\theta^+)}{p_1(\theta)} - \frac{p_s(\theta^+)}{p_s(\theta)}}{\frac{p_1(\theta^+)}{p_1(\theta)} - \frac{p_n(\theta^+)}{p_n(\theta)}} = \frac{\frac{p_1'(\theta)}{p_1(\theta)} - \frac{p_s'(\theta)}{p_s(\theta)}}{\frac{p_1'(\theta)}{p_1(\theta)} - \frac{p_n'(\theta)}{p_n(\theta)}}$$

and that:

$$\frac{d}{d\theta^+}\left(\frac{\frac{p_1(\theta^+)}{p_1(\theta)} - \frac{p_s(\theta^+)}{p_s(\theta)}}{\frac{p_1(\theta^+)}{p_1(\theta)} - \frac{p_n(\theta^+)}{p_n(\theta)}}\right) < 0, \ \forall\theta^+ > \theta$$

Hence, the term:

$$\sum_{s\neq 1,n} \pi_s(\theta)p_s(\theta)\left(\frac{\frac{p_1(\theta^+)}{p_1(\theta)} - \frac{p_s(\theta^+)}{p_s(\theta)}}{\frac{p_1(\theta^+)}{p_1(\theta)} - \frac{p_n(\theta^+)}{p_n(\theta)}} - \frac{\frac{p_1'(\theta)}{p_1(\theta)} - \frac{p_s'(\theta)}{p_s(\theta)}}{\frac{p_1'(\theta)}{p_1(\theta)} - \frac{p_n'(\theta)}{p_n(\theta)}}\right)$$

on the right-hand side of (26) is raised as $\pi_s(\theta)$ is decreased, so that (26) is relaxed. Also recall that:

$$\lim_{\theta^-\to\theta} \frac{\frac{p_1(\theta^-)}{p_1(\theta)} - \frac{p_s(\theta^-)}{p_s(\theta)}}{\frac{p_1(\theta^-)}{p_1(\theta)} - \frac{p_n(\theta^-)}{p_n(\theta)}} = \frac{\frac{p_1'(\theta)}{p_1(\theta)} - \frac{p_s'(\theta)}{p_s(\theta)}}{\frac{p_1'(\theta)}{p_1(\theta)} - \frac{p_n'(\theta)}{p_n(\theta)}},$$

and that:

$$\frac{d}{d\theta^-}\left(\frac{\frac{p_1(\theta^-)}{p_1(\theta)} - \frac{p_s(\theta^-)}{p_s(\theta)}}{\frac{p_1(\theta^-)}{p_1(\theta)} - \frac{p_n(\theta^-)}{p_n(\theta)}}\right) < 0, \ \forall \theta^- < \theta.$$

Hence, also the term:

$$\sum_{s\neq 1,n} \pi_s(\theta)\, p_s(\theta) \left(\frac{\frac{p_1(\theta^-)}{p_1(\theta)} - \frac{p_s(\theta^-)}{p_s(\theta)}}{\frac{p_1(\theta^-)}{p_1(\theta)} - \frac{p_n(\theta^-)}{p_n(\theta)}} - \frac{\frac{p_1'(\theta)}{p_1(\theta)} - \frac{p_s'(\theta)}{p_s(\theta)}}{\frac{p_1'(\theta)}{p_1(\theta)} - \frac{p_n'(\theta)}{p_n(\theta)}}\right)$$

in the right-hand side of (25) is raised as $\pi_s(\theta)$ is decreased, so that (25) is tightened.

G.4   Proof of Corollary 3

Condition (28) is satisfied if $\rho_s(\theta',\theta) < \rho_1(\theta',\theta) < \rho_n(\theta',\theta)$. We shall now consider cases in which one of these inequalities is violated.

First suppose that $\rho_s(\theta',\theta) > \rho_1(\theta',\theta)$ and $\rho_n(\theta',\theta) > \rho_1(\theta',\theta)$ for $\theta' \neq \theta$. Using these inequalities first for $\theta' = \theta^+$ and then for $\theta' = \theta^-$, we rewrite (28) as:

$$\frac{\frac{p_1(\theta^+)}{p_1(\theta)} - \frac{p_n(\theta^+)}{p_n(\theta)}}{\frac{p_1(\theta^+)}{p_1(\theta)} - \frac{p_s(\theta^+)}{p_s(\theta)}} < \frac{\rho_n(\theta^+,\theta) - \rho_1(\theta^+,\theta)}{\rho_s(\theta^+,\theta) - \rho_1(\theta^+,\theta)}$$

and as:

$$\frac{\frac{p_n(\theta^-)}{p_n(\theta)} - \frac{p_1(\theta^-)}{p_1(\theta)}}{\frac{p_s(\theta^-)}{p_s(\theta)} - \frac{p_1(\theta^-)}{p_1(\theta)}} < \frac{\rho_n(\theta^-,\theta) - \rho_1(\theta^-,\theta)}{\rho_s(\theta^-,\theta) - \rho_1(\theta^-,\theta)}.$$

In either inequality, the left-hand side is greater than 1. It is thus necessary that $\rho_n(\theta',\theta) > \rho_s(\theta',\theta)$ and that the difference $\rho_n(\theta',\theta) - \rho_s(\theta',\theta)$ be sufficiently large.

Next suppose that $\rho_1(\theta',\theta) > \rho_n(\theta',\theta)$ whereas $\rho_s(\theta',\theta) < \rho_1(\theta',\theta)$. Using these inequalities first for $\theta' = \theta^+$ and then for $\theta' = \theta^-$, we rewrite (28) as:

$$\frac{\frac{p_1(\theta^+)}{p_1(\theta)} - \frac{p_s(\theta^+)}{p_s(\theta)}}{\frac{p_1(\theta^+)}{p_1(\theta)} - \frac{p_n(\theta^+)}{p_n(\theta)}} < \frac{\rho_1(\theta^+,\theta) - \rho_s(\theta^+,\theta)}{\rho_1(\theta^+,\theta) - \rho_n(\theta^+,\theta)}$$

and as:

$$\frac{\frac{p_n(\theta^-)}{p_n(\theta)} - \frac{p_1(\theta^-)}{p_1(\theta)}}{\frac{p_s(\theta^-)}{p_s(\theta)} - \frac{p_1(\theta^-)}{p_1(\theta)}} > \frac{\rho_1(\theta^-,\theta) - \rho_n(\theta^-,\theta)}{\rho_1(\theta^-,\theta) - \rho_s(\theta^-,\theta)}.$$

The left-hand side in the former condition is lower than 1; the left-hand side in the latter condition is above 1. For these two conditions to hold, it is sufficient that $\rho_s(\theta',\theta) > \rho_n(\theta',\theta)$. It is necessary that the difference $\rho_n(\theta',\theta) - \rho_s(\theta',\theta)$ be not too large.

We are left with the case in which $\rho_n(\theta',\theta) < \rho_1(\theta',\theta) < \rho_s(\theta',\theta)$. We see that (28) is violated.

G.5 Proof of Corollary 4

Replacing $\pi_s(\theta) = -L$ in (27) and rearranging, (27) is rewritten as (36).

G.6 Proof of Corollary 5

Comparing (33) with (34), we see that (34) is tighter than (33) if and only if:

$$\frac{\frac{C(q(\theta),\theta)-C\left(q(\theta),\theta^-\right)}{\frac{p_n(\theta^-)}{p_n(\theta)}-\frac{p_1(\theta^-)}{p_1(\theta)}}-\frac{C\left(q(\theta),\theta^+\right)-C(q(\theta),\theta)}{\frac{p_1(\theta^+)}{p_1(\theta)}-\frac{p_n(\theta^+)}{p_n(\theta)}}}{-\sum_{s\neq 1,n}p_s(\theta)\left(\frac{\frac{p_1(\theta^+)}{p_1(\theta)}-\frac{p_s(\theta^+)}{p_s(\theta)}}{\frac{p_1(\theta^+)}{p_1(\theta)}-\frac{p_n(\theta^+)}{p_n(\theta)}}-\frac{\frac{p_1(\theta^-)}{p_1(\theta)}-\frac{p_s(\theta^-)}{p_s(\theta)}}{\frac{p_1(\theta^-)}{p_1(\theta)}-\frac{p_n(\theta^-)}{p_n(\theta)}}\right)} > \left(C\left(q(\theta),\theta\right)-C\left(q(\theta),\theta^-\right)\right)\frac{p_1(\theta)}{p_1(\theta)-p_1(\theta^-)}.$$

Let us group the terms including $\left(C\left(q(\theta),\theta\right)-C\left(q(\theta),\theta^-\right)\right)$ to rewrite:

$$\left(C\left(q(\theta),\theta\right)-C\left(q(\theta),\theta^-\right)\right)\left[\frac{1}{\frac{p_n(\theta^-)}{p_n(\theta)}-\frac{p_1(\theta^-)}{p_1(\theta)}}\right.$$

$$+\frac{p_1(\theta)}{p_1(\theta)-p_1(\theta^-)}\left(\left(\frac{\frac{p_1(\theta^+)}{p_1(\theta)}\sum_{s\neq 1,n}p_s(\theta)-\sum_{s\neq 1,n}p_s(\theta^+)}{\frac{p_1(\theta^+)}{p_1(\theta)}-\frac{p_n(\theta^+)}{p_n(\theta)}}\right)\right.$$

$$\left.\left.-\frac{\frac{p_1(\theta^-)}{p_1(\theta)}\sum_{s\neq 1,n}p_s(\theta)-\sum_{s\neq 1,n}p_s(\theta^-)}{\frac{p_1(\theta^-)}{p_1(\theta)}-\frac{p_n(\theta^-)}{p_n(\theta)}}\right)\right]$$

$$> \frac{C\left(q(\theta),\theta^+\right)-C\left(q(\theta),\theta\right)}{\frac{p_1(\theta^+)}{p_1(\theta)}-\frac{p_n(\theta^+)}{p_n(\theta)}}$$

Using $\sum_{s\neq 1,n}p_s(\cdot) = 1 - p_1(\cdot) - p_n(\cdot)$ and rearranging further yields:

$$\left(C\left(q(\theta),\theta\right)-C\left(q(\theta),\theta^-\right)\right)\frac{p_1(\theta)}{p_1(\theta)-p_1(\theta^-)}\left(p_n(\theta)+\frac{1-p_n(\theta)-\frac{p_1(\theta)}{p_1(\theta^+)}\left(1-p_n(\theta^+)\right)}{\frac{p_1(\theta)}{p_1(\theta^+)}\left(\frac{p_1(\theta^+)}{p_1(\theta)}-\frac{p_n(\theta^+)}{p_n(\theta)}\right)}\right)$$

$$> \frac{C\left(q(\theta),\theta^+\right)-C\left(q(\theta),\theta\right)}{\frac{p_1(\theta^+)}{p_1(\theta)}-\frac{p_n(\theta^+)}{p_n(\theta)}}. \tag{63}$$

We now take the expression in brackets in the left-hand side of (63) and factorize $p_n(\theta)$ to develop as follows:

$$p_n\left(\theta\right)\left(1+\frac{1-p_n(\theta)-\frac{p_1(\theta)}{p_1(\theta^+)}\left(1-p_n(\theta^+)\right)}{\frac{p_1(\theta)}{p_1(\theta^+)}\left(p_n\left(\theta\right)\frac{p_1(\theta^+)}{p_1(\theta)}-p_n(\theta^+)\right)}\right)$$

$$=\ p_n\left(\theta\right)\left(1+\frac{1-p_n(\theta)-\frac{p_1(\theta)}{p_1(\theta^+)}\left(1-p_n(\theta^+)\right)}{p_n\left(\theta\right)-p_1(\theta)\frac{p_n(\theta^+)}{p_1(\theta^+)}}\right)$$

$$=\ p_n(\theta)\frac{p_n(\theta)-p_1(\theta)\frac{p_n(\theta^+)}{p_1(\theta^+)}+1-p_n(\theta)-\frac{p_1(\theta)}{p_1(\theta^+)}\left(1-p_n(\theta^+)\right)}{p_n(\theta)-p_1(\theta)\frac{p_n(\theta^+)}{p_1(\theta^+)}}$$

$$=\ \frac{p_1(\theta^+)-p_1(\theta)}{p_1(\theta^+)-p_1(\theta)\frac{p_n(\theta^+)}{p_n(\theta)}}.$$

Using this, we can now rewrite (63) as:

$$\left(C\left(q\left(\theta\right),\theta\right)-C\left(q\left(\theta\right),\theta^-\right)\right)\frac{\frac{p_1(\theta^+)-p_1(\theta)}{p_1(\theta)-p_1(\theta^-)}}{\frac{p_1(\theta^+)}{p_1(\theta)}-\frac{p_n(\theta^+)}{p_n(\theta)}}>\frac{C\left(q\left(\theta\right),\theta^+\right)-C\left(q\left(\theta\right),\theta\right)}{\frac{p_1(\theta^+)}{p_1(\theta)}-\frac{p_n(\theta^+)}{p_n(\theta)}}$$

or, equivalently, as:

$$\frac{C\left(q\left(\theta\right),\theta\right)-C\left(q\left(\theta\right),\theta^-\right)}{C\left(q\left(\theta\right),\theta^+\right)-C\left(q\left(\theta\right),\theta\right)}>\frac{p_1(\theta)-p_1(\theta^-)}{p_1(\theta^+)-p_1(\theta)},$$

which means that (16) is violated. Therefore, (34) implies (33) if and only if (16) is violated.

H   Second best

H.1   Three types

Denote $f\left(\theta_t\right)$ the probability of P facing type $\theta_t\in\Theta_3$. Further denote $\gamma_s\left(\theta_t\right)$ the multiplier associated with (LL) when signal is s and type is θ_t, $\zeta\left(\theta_t\right)$ that associated with (PC) when type is θ_t, λ that associated with (37), μ that associated with (38), β that associated with (39),

δ that associated with (40. The Lagrangian of the programme is:

$$\sum_{\theta_t \in \Theta_3} [S(q(\theta_t)) - C(q(\theta_t), \theta_t) - R(\theta_t)] f(\theta_t) + \sum_{\theta_t \in \Theta_3} \sum_{s \in N} \gamma_s(\theta)(\pi_s(\theta_t) + L) + \sum_{\theta_t \in \Theta_3} \zeta(\theta_t) R(\theta_t)$$

$$+\lambda \left\{ \pi_n(\theta_2) p_n(\theta_2) \right.$$

$$-\frac{\frac{p_1(\theta_3)}{p_1(\theta_2)} R(\theta_2) - R(\theta_3) - [C(q(\theta_2), \theta_3) - C(q(\theta_2), \theta_2)] - \sum_{s \neq 1,n} \pi_s(\theta_2) p_s(\theta_2) \left(\frac{p_1(\theta_3)}{p_1(\theta_2)} - \frac{p_s(\theta_3)}{p_s(\theta_2)} \right)}{\frac{p_1(\theta_3)}{p_1(\theta_2)} - \frac{p_n(\theta_3)}{p_n(\theta_2)}} \right\}$$

$$+\mu \left\{ \frac{R(\theta_1) - \frac{p_1(\theta_1)}{p_1(\theta_2)} R(\theta_2) - [C(q(\theta_2), \theta_2) - C(q(\theta_2), \theta_1)] - \sum_{s \neq 1,n} \pi_s(\theta_2) p_s(\theta_2) \left(\frac{p_s(\theta_1)}{p_s(\theta_2)} - \frac{p_1(\theta_1)}{p_1(\theta_2)} \right)}{\frac{p_n(\theta_1)}{p_n(\theta_2)} - \frac{p_1(\theta_1)}{p_1(\theta_2)}} \right.$$

$$\left. -\pi_n(\theta_2) p_n(\theta_2) \right\}$$

$$+\beta \frac{p_1(\theta_2)}{p_1(\theta_3)} \left\{ \frac{p_1(\theta_3)}{p_1(\theta_2)} R(\theta_2) - R(\theta_3) - \frac{p_1(\theta_3)}{p_1(\theta_2)} [C(q(\theta_3), \theta_3) - C(q(\theta_3), \theta_2)] \right.$$

$$\left. -\frac{p_1(\theta_3)}{p_1(\theta_2)} \sum_{s \neq 1} \pi_s(\theta_3) p_s(\theta_3) \left(\frac{p_s(\theta_2)}{p_s(\theta_3)} - \frac{p_1(\theta_2)}{p_1(\theta_3)} \right) \right\}$$

$$+\delta \frac{p_1(\theta_2)}{p_1(\theta_1)} \left\{ \frac{p_1(\theta_1)}{p_1(\theta_2)} R(\theta_2) - R(\theta_1) + \frac{p_1(\theta_1)}{p_1(\theta_2)} [C(q(\theta_1), \theta_2) - C(q(\theta_1), \theta_1)] \right.$$

$$\left. +\frac{p_1(\theta_1)}{p_1(\theta_2)} \sum_{s \neq 1} \pi_s(\theta_1) p_s(\theta_1) \left(\frac{p_1(\theta_2)}{p_1(\theta_1)} - \frac{p_s(\theta_2)}{p_s(\theta_1)} \right) \right\}.$$

We now characterize the solution.

First suppose that $\zeta(\theta_1) = \zeta(\theta_2) = 0$. The Lagrangian is linear in both $\frac{p_1(\theta_3)}{p_1(\theta_2)} R(\theta_2) - R(\theta_3)$ and $R(\theta_1) - \frac{p_1(\theta_1)}{p_1(\theta_2)} R(\theta_2)$, with coefficients:

$$\beta \frac{p_1(\theta_2)}{p_1(\theta_3)} - \frac{\lambda}{\frac{p_1(\theta_3)}{p_1(\theta_2)} - \frac{p_n(\theta_3)}{p_n(\theta_2)}}$$

$$\frac{\mu}{\frac{p_n(\theta_1)}{p_n(\theta_2)} - \frac{p_1(\theta_1)}{p_1(\theta_2)}} - \delta \frac{p_1(\theta_2)}{p_1(\theta_1)}.$$

Suppose that $\beta = 0$. Then, the former coefficient is negative, and hence $\frac{p_1(\theta_3)}{p_1(\theta_2)} R(\theta_2) - R(\theta_3)$ should be decreased until the point where the constraint with β is binding. Then, $\beta > 0$, in contradiction with the hypothesis that $\beta = 0$. Suppose that $\delta = 0$. Then, the latter coefficient is positive, and hence $R(\theta_1) - \frac{p_1(\theta_1)}{p_1(\theta_2)} R(\theta_2)$ should be increased until the point where $\delta > 0$, which contradicts the hypothesis that $\delta = 0$. We thus conclude that if $\zeta(\theta_1) = \zeta(\theta_2) = 0$, then both $\beta > 0$ and $\delta > 0$. Next suppose that $\zeta(\theta_1) > 0$ and $\zeta(\theta_2) > 0$. It is immediate to see that $\beta = 0$ and $\delta = 0$.

We now turn to show that $\gamma_n(\theta_2) = 0$ is equivalent to $\beta > 0$ and $\delta > 0$, and hence it is equivalent to $\zeta(\theta_1) > 0$ and $\zeta(\theta_2) > 0$.

Suppose that $\gamma_n(\theta_2) = 0$. The Lagrangian is linear in $\pi_n(\theta_2)$ with coefficient $(\lambda - \mu) p_n(\theta_2)$. If $\lambda > \mu = 0$, then the Lagrangian increases with $\pi_n(\theta_2)$. Hence, $\pi_n(\theta_2)$ should be raised until the point where $\mu > 0$, in contradiction with the hypothesis that $\mu = 0$. Analogous contradiction emerges if we suppose that $\mu > \lambda = 0$. Provided that at second best it cannot be $\lambda = \mu = 0$ (as (37) and (38) do not hold jointly at the first-best allocation), it must be the case that $\lambda > 0$ and $\mu > 0$. Suppose that $\lambda \neq \mu$. As the two constraints with these multipliers are binding, it

must be the case that $(\lambda - \mu)\,\pi_n\,(\theta_2)\,p_n(\theta_2) = 0$. However, if $\lambda \neq \mu$, then the Lagrangian either increases or decreases with $\pi_n\,(\theta_2)$, involving that it should be $\pi_n\,(\theta_2) \neq 0$, in contradiction with the requirement that $(\lambda - \mu)\,\pi_n\,(\theta_2)\,p_n(\theta_2) = 0$. We conclude that $\lambda = \mu$.

The Lagrangian is linear in $\pi_s\,(\theta_2)$ with the following coefficient:

$$
p_s(\theta_2)\left(\lambda \frac{\frac{p_1(\theta_3)}{p_1(\theta_2)} - \frac{p_s(\theta_3)}{p_s(\theta_2)}}{\frac{p_1(\theta_3)}{p_1(\theta_2)} - \frac{p_n(\theta_3)}{p_n(\theta_2)}} - \mu \frac{\frac{p_s(\theta_1)}{p_s(\theta_2)} - \frac{p_1(\theta_1)}{p_1(\theta_2)}}{\frac{p_n(\theta_1)}{p_n(\theta_2)} - \frac{p_1(\theta_1)}{p_1(\theta_2)}} \right)
$$
$$
= \; p_s(\theta_2)\lambda \left(\frac{\frac{p_1(\theta_3)}{p_1(\theta_2)} - \frac{p_s(\theta_3)}{p_s(\theta_2)}}{\frac{p_1(\theta_3)}{p_1(\theta_2)} - \frac{p_n(\theta_3)}{p_n(\theta_2)}} - \frac{\frac{p_s(\theta_1)}{p_s(\theta_2)} - \frac{p_1(\theta_1)}{p_1(\theta_2)}}{\frac{p_n(\theta_1)}{p_n(\theta_2)} - \frac{p_1(\theta_1)}{p_1(\theta_2)}} \right).
$$

Relying on (35), this is found to be negative, involving that $\gamma_s\,(\theta_2) > 0$ and $\pi_s\,(\theta_2) = -L$.

We now verify the hypothesis that $\gamma_n\,(\theta_2) = 0$. Resting on the binding constraints with λ and μ, we see that $\pi_n\,(\theta_2) > -L$ if and only if these two conditions are both satisfied:

$$
\frac{\frac{p_1(\theta_3)}{p_1(\theta_2)} R\,(\theta_2) - R\,(\theta_3) - [C\,(q\,(\theta_2),\theta_3) - C\,(q\,(\theta_2),\theta_2)] - \sum_{s \neq 1,n} \pi_s\,(\theta_2)\,p_s(\theta_2) \left(\frac{p_1(\theta_3)}{p_1(\theta_2)} - \frac{p_s(\theta_3)}{p_s(\theta_2)} \right)}{\frac{p_1(\theta_3)}{p_1(\theta_2)} - \frac{p_n(\theta_3)}{p_n(\theta_2)}}
$$
$$
\geq \; -L p_n(\theta_2)
$$
$$
\frac{R\,(\theta_1) - \frac{p_1(\theta_1)}{p_1(\theta_2)} R\,(\theta_2) - [C\,(q\,(\theta_2),\theta_2) - C\,(q\,(\theta_2),\theta_1)] - \sum_{s \neq 1,n} \pi_s\,(\theta_2)\,p_s(\theta_2) \left(\frac{p_s(\theta_1)}{p_s(\theta_2)} - \frac{p_1(\theta_1)}{p_1(\theta_2)} \right)}{\frac{p_n(\theta_1)}{p_n(\theta_2)} - \frac{p_1(\theta_1)}{p_1(\theta_2)}}
$$
$$
\geq \; -L p_n(\theta_2)
$$

With $\pi_s\,(\theta_2) = -L$, $\forall s \neq 1, n$, these conditions are the same as the constraints with β and δ. Hence, if $\gamma_n\,(\theta_2) = 0$ and so $\pi_n\,(\theta_2) > -L$, then the constraints with β and δ are slack, in which case $\beta = 0$ and $\delta = 0$, further involving that $\zeta\,(\theta_1) > 0$ and $\zeta\,(\theta_2) > 0$. If $\gamma_n\,(\theta_2) > 0$ and so $\pi_n\,(\theta_2) = -L$, then $\beta > 0$ and $\delta > 0$, in which case $\zeta\,(\theta_1) = \zeta\,(\theta_2) = 0$.

Therefore, there are two solutions. The first applies when $\gamma_n\,(\theta_2) > 0$, $\beta > 0$, $\delta > 0$ and $\zeta\,(\theta_1) = \zeta\,(\theta_2) = 0$. From the constraints with β and δ, we find:

$$
R\,(\theta_2) \;=\; \frac{p_1\,(\theta_2)}{p_1(\theta_3)} R\,(\theta_3) + [C\,(q\,(\theta_3),\theta_3) - C\,(q\,(\theta_3),\theta_2)] \tag{64}
$$
$$
+ \sum_{s \neq 1} \pi_s\,(\theta_3)\,p_s(\theta_3) \left(\frac{p_s\,(\theta_2)}{p_s(\theta_3)} - \frac{p_1\,(\theta_2)}{p_1\,(\theta_3)} \right)
$$
$$
R\,(\theta_1) \;=\; \frac{p_1(\theta_1)}{p_1\,(\theta_2)} R\,(\theta_2) + \frac{p_1(\theta_1)}{p_1\,(\theta_2)} [C\,(q\,(\theta_1),\theta_2) - C\,(q\,(\theta_1),\theta_1)] \tag{65}
$$
$$
+ \frac{p_1(\theta_1)}{p_1\,(\theta_2)} \sum_{s \neq 1} \pi_s\,(\theta_1)\,p_s(\theta_1) \left(\frac{p_1\,(\theta_2)}{p_1\,(\theta_1)} - \frac{p_s\,(\theta_2)}{p_s(\theta_1)} \right).
$$

Replacing in the Lagrangian, we see that P should set $R\,(\theta_3) = 0$ together with $\pi_s\,(\theta_1) = \pi_s\,(\theta_3) = -L$ $\forall s \neq 1$. Replacing $R\,(\theta_3) = 0$ and $\pi_s\,(\theta_3) = -L$ $\forall s \neq 1, n$ in (64) yields (42). Replacing the obtained value of $R\,(\theta_2)$ and $\pi_s\,(\theta_1) = -L$ $\forall s \neq 1, n$ in (65) yields (41).

The second solution applies when $\gamma_n\,(\theta_2) = 0$, $\zeta\,(\theta_1) > 0$, $\zeta\,(\theta_2) > 0$ and $\beta = \delta = 0$. Then, $R\,(\theta_1) = R\,(\theta_2) = 0$. Replacing these values, $R\,(\theta_3) = 0$ and $\pi_s\,(\theta_2) = -L$ $\forall s \neq 1, n$ in (37) and (38), we obtain the necessary condition (34) where $\theta^- = \theta_1$, $\theta = \theta_2$, $\theta^+ = \theta_3$. This condition is binding because $\lambda > 0$ as well as $\mu > 0$.

H.2   A continuum of types. Proof of Proposition 2

We first show that (LIC) is rewritten as (44).

The incentive constraint whereby type θ^- is unwilling to report θ is:

$$R\left(\theta\right) \leq \frac{p_1\left(\theta\right)}{p_1\left(\theta^-\right)} R\left(\theta^-\right) - \frac{p_1\left(\theta\right)}{p_1\left(\theta^-\right)} \left[C\left(q\left(\theta\right),\theta\right) - C\left(q\left(\theta\right),\theta^-\right)\right]$$
$$- \sum_{s\neq 1} \pi_s\left(\theta\right) p_s\left(\theta^-\right) \left(\frac{p_1\left(\theta\right)}{p_1\left(\theta^-\right)} - \frac{p_s\left(\theta\right)}{p_s\left(\theta^-\right)}\right).$$

This is rewritten as:

$$R\left(\theta\right) - R\left(\theta^-\right) \leq \frac{p_1\left(\theta\right) - p_1\left(\theta^-\right)}{p_1\left(\theta\right)} R\left(\theta\right) - \left[C\left(q\left(\theta\right),\theta\right) - C\left(q\left(\theta\right),\theta^-\right)\right]$$
$$+ \sum_{s\neq 1} \pi_s\left(\theta\right) p_s(\theta) \left(\frac{p_s(\theta) - p_s\left(\theta^-\right)}{p_s(\theta)} - \frac{p_1(\theta) - p_1\left(\theta^-\right)}{p_1(\theta)}\right).$$

Divide all terms by $\theta - \theta^- > 0$ and take the limit for $\theta^- \to \theta$ to obtain:

$$R'\left(\theta\right) \leq \frac{p_1'(\theta)}{p_1(\theta)} R\left(\theta\right) - C_\theta\left(q\left(\theta\right),\theta\right) + \sum_{s\neq 1} \pi_s\left(\theta\right) p_s(\theta) \left(\frac{p_s'\left(\theta\right)}{p_s(\theta)} - \frac{p_1'\left(\theta\right)}{p_1(\theta)}\right). \tag{66}$$

The incentive constraint whereby type θ^+ is unwilling to report θ is:

$$\frac{p_1\left(\theta^+\right)}{p_1(\theta)} R\left(\theta\right) \leq R\left(\theta^+\right) - \left[C\left(q\left(\theta\right),\theta\right) - C\left(q\left(\theta\right),\theta^+\right)\right]$$
$$- \sum_{s\neq 1} \pi_s\left(\theta\right) p_s(\theta) \left(\frac{p_s\left(\theta^+\right)}{p_s(\theta)} - \frac{p_1\left(\theta^+\right)}{p_1\left(\theta\right)}\right).$$

This is rewritten as:

$$R\left(\theta^+\right) - R\left(\theta\right) \geq \frac{p_1\left(\theta^+\right) - p_1(\theta)}{p_1(\theta)} R\left(\theta\right) - \left[C\left(q\left(\theta\right),\theta^+\right) - C\left(q\left(\theta\right),\theta\right)\right]$$
$$+ \sum_{s\neq 1} \pi_s\left(\theta\right) p_s(\theta) \left(\frac{p_s(\theta^+) - p_s(\theta)}{p_s(\theta)} - \frac{p_1(\theta^+) - p_1\left(\theta\right)}{p_1\left(\theta\right)}\right).$$

Divide all terms by $\theta^+ - \theta > 0$ and take the limit for $\theta^+ \to \theta$ to obtain:

$$R'\left(\theta\right) \geq \frac{p_1'\left(\theta\right)}{p_1(\theta)} R\left(\theta\right) - C_\theta\left(q\left(\theta\right),\theta\right) + \sum_{s\neq 1} \pi_s\left(\theta\right) p_s(\theta) \left(\frac{p_s'(\theta)}{p_s(\theta)} - \frac{p_1'(\theta)}{p_1\left(\theta\right)}\right). \tag{67}$$

Putting together (66) and (67) yields (43).

Rearrange (43) to obtain:

$$R'\left(\theta\right) - \frac{p_1'(\theta)}{p_1(\theta)} R\left(\theta\right) = \sum_{s\neq 1} \pi_s\left(\theta\right) p_s(\theta) \left(\frac{p_s'\left(\theta\right)}{p_s(\theta)} - \frac{p_1'\left(\theta\right)}{p_1(\theta)}\right) - C_\theta\left(q\left(\theta\right),\theta\right).$$

This is a first-degree differential equation in the generic form:

$$R'(\theta) + x(\theta) R(\theta) = y(\theta),$$

where

$$
\begin{aligned}
x(\theta) &= -\frac{p_1'(\theta)}{p_1(\theta)} \\
y(\theta) &= \sum_{s=2}^{n} \pi_s(\theta) p_s(\theta) \left(\frac{p_s'(\theta)}{p_s(\theta)} - \frac{p_1'(\theta)}{p_1(\theta)} \right) - C_\theta(q(\theta),\theta).
\end{aligned}
$$

The integrating factor is

$$\mu(\theta) = e^{\int -\frac{p_1'(\theta)}{p_1(\theta)} d\theta} = e^{-\ln(p_1(\theta))} = \frac{1}{e^{\ln(p_1(\theta))}} = \frac{1}{p_1(\theta)}$$

Multiply either side of the differential equation by $\mu(\theta)$ to obtain:

$$
\begin{aligned}
&\left(R'(\theta) - \frac{p_1'(\theta)}{p_1(\theta)} R(\theta) \right) \frac{1}{p_1(\theta)} \\
&= \left[\sum_{s \neq 1} \pi_s(\theta) p_s(\theta) \left(\frac{p_s'(\theta)}{p_s(\theta)} - \frac{p_1'(\theta)}{p_1(\theta)} \right) - C_\theta(q(\theta),\theta) \right] \frac{1}{p_1(\theta)}
\end{aligned}
$$

Using:

$$\frac{1}{p_1(\theta)} \left(R'(\theta) - R(\theta) \frac{p_1'(\theta)}{p_1(\theta)} \right) = \left(\frac{R(\theta)}{p_1(\theta)} \right)',$$

the differential equation is rewritten as:

$$\left(\frac{R(\theta)}{p_1(\theta)} \right)' = \left[\sum_{s \neq 1} \pi_s(\theta) p_s(\theta) \left(\frac{p_s'(\theta)}{p_s(\theta)} - \frac{p_1'(\theta)}{p_1(\theta)} \right) - C_\theta(q(\theta),\theta) \right] \frac{1}{p_1(\theta)}.$$

Integrating from $\bar{\theta}$ to θ yields:

$$
\begin{aligned}
R(\theta) =\ & R(\bar{\theta}) \\
& + \int_{\bar{\theta}}^{\theta} \left[\sum_{s \neq 1} \pi_s(x) p_s(x) \left(\frac{p_s'(x)}{p_s(x)} - \frac{p_1'(x)}{p_1(x)} \right) - C_x(q(x),x) \right] \frac{p_1(\theta)}{p_1(x)} dx,
\end{aligned}
$$

which is rewritten as (44).

Take any triplet of types $\{\theta^-, \theta, \theta^+\}$ such that $\theta^- < \theta < \theta^+$. Denote $\lambda(\theta,\theta^+)$, $\mu(\theta,\theta^-)$, $\beta(\theta,\theta^+)$ and $\delta(\theta,\theta^-)$ the multipliers respectively associated with $IC_{\theta^+}^\theta$, $IC_{\theta^-}^\theta$, $IC_\theta^{\theta^+}$ and $IC_\theta^{\theta^-}$, where $IC_{\theta^+}^\theta$ is the incentive constraint whereby type θ^+ is unwilling to report θ, and analogously for the others. Proceeding in a similar fashion to the three-type case, one can assess that $\lambda(\theta,\theta^+) = \mu(\theta,\theta^-) > 0$ and that $R(\theta) = 0$ when $\gamma_n(\theta) = 0$. The results so obtained hold true for all $\theta \in (\underline{\theta}, \bar{\theta})$.

The constraints associated with $\lambda(\theta,\theta^+)$ and $\mu(\theta,\theta^-)$ are binding. Resting on this and

replacing $R\left(\cdot\right) = 0$, we obtain:

$$\frac{\frac{C(q(\theta),\theta)-C\left(q(\theta),\theta^-\right)}{\frac{p_n(\theta^-)}{p_n(\theta)}-\frac{p_1(\theta^-)}{p_1(\theta)}} - \frac{C\left(q(\theta),\theta^+\right)-C(q(\theta),\theta)}{\frac{p_1(\theta^+)}{p_1(\theta)}-\frac{p_n(\theta^+)}{p_n(\theta)}}}{-\sum_{s\neq 1,n} p_s(\theta)\left(\frac{\frac{p_1(\theta^+)}{p_1(\theta)}-\frac{p_s(\theta^+)}{p_s(\theta)}}{\frac{p_1(\theta^+)}{p_1(\theta)}-\frac{p_n(\theta^+)}{p_n(\theta)}} - \frac{\frac{p_1(\theta^-)}{p_1(\theta)}-\frac{p_s(\theta^-)}{p_s(\theta)}}{\frac{p_1(\theta^-)}{p_1(\theta)}-\frac{p_n(\theta^-)}{p_n(\theta)}}\right)} = L.$$

Provided that (34) does not hold at the first-best level of $q\left(\theta\right)$, this condition is satisfied by decreasing the value of the difference:

$$\frac{C\left(q\left(\theta\right),\theta\right) - C\left(q\left(\theta\right),\theta^-\right)}{\frac{p_n(\theta^-)}{p_n(\theta)} - \frac{p_1(\theta^-)}{p_1(\theta)}} - \frac{C\left(q\left(\theta\right),\theta^+\right) - C\left(q\left(\theta\right),\theta\right)}{\frac{p_1(\theta^+)}{p_1(\theta)} - \frac{p_n(\theta^+)}{p_n(\theta)}},$$

which requires distorting $q\left(\theta\right)$ upwards, provided that $C_{q\theta} > 0$.