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The Free Solidarity Value

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The Free Solidarity Value

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Abstract

In this paper, we consider a cooperative game in which two types of players co-exist: solidary and non solidary players. Solidary players are able to support by consent at least one of their weaker partners without disadvantaging non-solidary players. We present a value of the game which takes ito account the types of players and satisfies some appropriate axioms: Efficiency, Additivity, Symmetry among players who have the same type, Conditional null player, and a new axiom, the Unaffected Allocation of non-solidary players - **(UA)** - which is defined as follows: when players have the possibility to decide freely to be solidary, this should affect neither the allocations of non-solidary players, nor the outcome of the game."

Keywords: Shapley value, Nowak and Radzik value, the free solidarity value, TU cooperative games.
Classification-JEL : C71, D60

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1 Introduction

In cooperative games, the most difficult economic problem is how of gain from cooperation is shared. A pioneering study that addresses this issue is that of Shapley [1953]. He considers a superadditive cooperative game in characteristic function form, and proposes as a solution to distribute "fairly" the gains of the coalition of players. This result has many applications, particularly in economics, and in social and political science. For each player, the famous Shapley value assigns it's expected marginal contribution with respect to a uniform distribution over the set of all permutations on the set of players. Remarkably Shapley proved that the proposed value is the unique value which satisfies four acceptable axioms. The first one (Efficiency) says that players share among themselves the resources available to the grand coalition. The second one (Additivity) requires that the value be additive on the space of all games. The third one (Symmetry) requires that players who make the same marginal contribution to any coalition, should have the same value. The last one (Null player) requires that players who has zero marginal contribution to any coalition must have zero payoffs.

Most of contributions in literature are devoted to alternative axiomatic characterizations of the Shapley value, like Hart and Mas-Colell's axiom of consistency, Young's axiom of monotonicity and the notion of potential Winter (1992). When some partition of players is given a priori other than the grand coalition, Aumann and Drèze (1974) and Owen (1977) generalize the Shapley value in this environment. The extensions to NTU (non transferable utility) games and non atomic games can be found in Aumann and Shapley (1974) and in Winter (1991). Roth (1977) showed that the Shapley value can be expressed as a von Neumann-Morgenstern utility function. For the uniqueness of the Shapley value, when we restrict the class of all games to the context of the subclass of games, one can see Dubey (1975) and Neyman (1989).

We will return to the Shapley axioms, especially the axiom of zero player who said "players who have no marginal contribution in any coalition must have zero payoffs". This is a difficult task because, for the survival of a group, it is necessary to have some gain transfer to that specific player. In our societies there is often a gain transfer to people in unemployment, it offers them to lead a dignified life and also allow them to have new skills to meet labor supply market. Societies also support disabled peoples with a minimum income and training appropriated to their disability, so that they can integrate the society and live normally. Thus, we speak about solidarity. The solidarity is a personal (or a group) value. In social group, solidarity is the capacity of performing as a whole in a group. It denotes a high degree of integration and internal stability, which implies to assume and to share the benefits and the risks.

Nowak and Radzik (1994) have proposed a new solution for cooperative TU games. This value is called the solidarity value.[1] This solution concept, which gives

[1] The solidarity value was first introduced by Sprumont (1990), in a recursive way. Calvo (2008) has shown that both definitions are equivalent.

for each game one solution, is based on the Shapley value. Formally, the Nowak and Radzik (1994). value appears as a weighted sum of the average contributions of different coalitions of a player.

If a player wants to join a coalition, he will get a fraction which depends on the average contribution of players in this coalition. If the marginal contribution of the player is higher than the average contribution of the coalition, the latter will offer a part of his contribution to the coalition and thus help those whose marginal contribution does not reach the average contribution of the coalition. Hence the use of the word "solidarity" by Nowak and Radzik.

Calvoy and Gutiérrez [2010] have looked to the two-step Shapley value, a value introduced by Kamijo (2009), by explicitly introducing this solidarity principle in the axiomatic system. This yields additional support for the two step Shapley value as an interesting alternative to the Owen value whenever solidarity matters.

In a recent work Calvoy and Gutiérrez (2014), authors define and characterize the class of all weighted solidarity values. Their characterization employs the classical axioms determining the solidarity value (except symmetry), that is, efficiency, additivity and the A-null player axiom, and two new axioms proportionality and strong individual rationality. In is shown in a second axiomatization, that the additivity and the A-null player axioms are replaced by a new axiom called average marginality.

The limit of these techniques is that they assumes that, in a coalition, all the players with a marginal contribution (which is) higher than the average contribution of the coalitions, *must* behave in solidarity with the others. Secondly the solidarity value can lead to situations where the poor (a player who is a null player or has a small value) through transfers can achieve a more favorable status, some times he (or she) can be in a social level near to those players who participated in solidarity. As we see in the life of every day the solidarity is intended to keep people in a social level so that they can improve their standard of living by looking for better job opportunities, making training to adapt to the supply of labor market .... etc ...

The principle of solidarity, as we conceive in this work, differs from the various measures of solidarity offered so far, by the fact that it is offered and natural: the player decides to give, *with his consent*, and without expecting the slightest compensation for his contribution, thus it is a purely human act (without obligation). Each player can give a fraction of his own payoff and favor one or several players without disadvantaging other players in the game. This solidarity is thus far from being compulsory and strategic; it is a natural and free act.

The fact that players decide, naturally, to behave as solidary players (and are not forced to do so) does not diminish the importance of the following question: how to share the gains in a cooperative game in the presence of a coalition of solidary players? In other words, how can we calculate the new value denoted by $\zeta(N, v, S^*)$ in this work?

The rest of the paper is organized as follows. Section 2 is devoted to basic definitions. Section 3 introduces the free solidarity value. This is done with the help of illustrative example designed to show that, when the solidarity is free, it can improve the result of the game. We provide the axiomatic characterization of this value in Section 4. The conclusions are presented in Section 5.

2 The Shapley value, The Solidarity value

A coalitional form game with transferable utility (every coalition can divide its worth in any possible way among its members) on a finite set of players $N = \{1, 2...n\}$ is a function v from the set of all coalitions 2^N to $\mathbb{R}$, and assigns to each coalition $S \subseteq N$ a real value in $\mathbb{R}$ with $v(\varnothing) = 0$. The real $v(S)$ represents the total payoff or rent the coalition S can get in the game v and it is considered as a monetary value (players having similar preferences) in the case of cooperative TU-games. A coalition S is a subset of the set $N = \{1, 2...n\}$ of n players, with $S \subseteq N$. $S = \{i\}$ is a coalition of one player (singleton), $S = \{N\}$ is a coalition of all the players in the game (grand coalition). For any coalition S or T, let denote by s respectively t the cardinal (the number of players) of S respectively of T, some times for a coalition S, we write its cardinality by $\mid S \mid$ instead of s.

For a game (N, v), a value is a function ψ which associates to each player $i \in N$ a real number $\psi_i(N, v)$ which represents the payoff of player i when he participates in the game (N, v).

The famous Shapley value assigns an expected marginal contribution to each player in the game with respect to a uniform distribution over the set of all permutations on the set of players. Shapley proved that the proposed value is the unique value which satisfy four acceptable axioms. The first (Efficiency) says that players distribute among themselves the resources available to the grand coalition.

Axiom A1 (Efficiency): For any game $(N, v) \in G^N$, $\sum\limits_{i \in N} \psi_i = v(N)$; where $v(N)$ is a value of the grand coalition.

The second (Additivity) requires that the value be an additive on the space of all games.

Axiom A2 (Additivity): For any games (N, v), $(N, w) \in G^N$, $\psi(N, v + w) = \psi(N, v) + \psi(N, w)$. This axiom, which clarifies how the values of different games must be related to one another, is the ingenious idea behind Shapley's demonstration.

The third (Symmetry) requires: Players who make the same marginal contribution to any coalition, have the same value.

Axiom A3 (Symmetry): If $\forall S \subset N$, $i, j \notin S, v(S \cup \{i\}) = v(S \cup \{j\})$, then $\psi_i = \psi_j$. i, j are said to be symmetrical players. In this axiom the names of the players play no role in determining the value.

The last (Null player) requires: players whose marginal contribution is null with respect to every coalition must have zero payoffs.

Axiom A4 (Null player): A player $i \in N$ is qualified as a null-player if his contributions, without exception, are 0 , i.e., $\psi_i(N, v) = 0$.

The Shapley value depends on the marginal contribution of players in all the coalitions he can join. Let $C_i(v, S)$ be the marginal contribution of player i in the coalition $S \subseteq N$ in the game (N, v). This contribution can be formally defined as follows:

$$C_i(v, S) = v(S) - v(S/\{i\}).$$

The Shapley value $Sh(N, v) \in \mathbb{R}^{\mathbb{N}}$ is given by:

$$Sh_i(N, v) = \sum_{S \ni i} \frac{(n-s)!(s-1)!}{n!}[v(S) - v(S \backslash \{i\}] \tag{1}$$

Theorem 1 [Shapley 1953] *A value ψ on G^N satisfies efficiency, additivity, symmetry and the null player axiom, if and only if ψ is the Shapleyvalue, i.e., $\psi(N, v) = Sh(N, v)$.*

The layout of the proof that Shapley elaborated consists to think of a characteristic function v as a vector with $2^n - 1$ components expressed in a space generated by $2^n - 1$ unanimity games w_T. Then the set of all games in characteristic form is exactly the Euclidean space of dimension $2^n - 1$. If we know the value of each components (which is a game) of the basis, then we can determine the value for any game. The additivity axiom is determining to elaborates this result. The basis is defined from the unanimity games defined by for fixed coalition T, $w_T(S) = \begin{cases} 1, & T \subseteq S \\ 0, & T \not\subset S \end{cases}$ .If for some player $i \notin T$, then i is a null player in this game, which is obvious from the fact that the coalition S containing i are such that $T \not\subset S$ and the $W_T(S) = 0$, then from null player axiom player i gets zero. Players in T are symmetric, they get the same value (from symmetric axiom). From efficiency players in T get $\frac{1}{t}$ as value. A useful property of the unanimity games $w_T(.)$ when T goes through the different coalitions of N is to form a basis for the set of all games. Therefore any game can be written as the sum of $\alpha_T w_T$. The axiom of additivity completes the proof. [2]

In Nowak and Radzik (1994) have proposed a value named solidarity value. This new value attributes to any player $i \in N$ the average marginal contribution of all players in the same coalition $S \subseteq N$. Let $C^{av}(S)$ be the average contribution of coalition S in the game (N, v). In this case, all the players forming S have $C^{av}(S)$ as their contribution.

$$C^{av}(S) = \tfrac{1}{s}\sum_{i \in S}[v(S) - v(S \backslash \{i\}] = \tfrac{1}{s}\sum_{i \in S} C_i(v, S)$$

The Solidarity value of a TU-game (N, v) is a unique vector (n-tuple) $NR(N, v) \in \mathbb{R}^n$, calculated as follows:

$$NR_i(N, v) = \sum_{S \ni i} \frac{(n-s)!(s-1)!}{n!}[C^{av}(S)] \tag{2}$$

[2]Certainly a useful reference for Shapley value and the different extensions is Roth, A. E. (1988).

Compared to Shapley (1953),The only new axiom introduced by Nowak and Radzik (1994) is the A-null player axiom.

If it happens that every coalition S containing player i has a null average contribution then player i gets nothing from the game v.

A-null player: If $C^{av}(S) = 0$ for every coalition S containing i, then $NR_i(N, v) = 0$. Player i is said to be an A-null player

Theorem 2 [Nowak and Radzik,1994] *A value $\psi \in G^N$ satisfies the efficiency, additivity, symmetry and A-null player axioms if and only if $\psi(N, v) = NR(N, v)$, i.e., ψ is the solidarity value.*

It is clear that both values only differ in the treatment of the null players. The null player axiom says that if all the marginal contributions of a player in a game are zero, then the player should obtain zero. The interpretation of the A-null player axiom is less evident. Notice that $C^{av}(S) = 0$ means that the expected productivity of the players in coalition S is zero. The A-null player axiom says that when the average productivity of all coalitions to which the player belongs are zero then he *must receive* zero.

Let's start with an example through which we will see how to use both seminal methods of resolution, i.e., the Shapley value and the Nowak and Radzik value.

Example 1 (Three Brothers):[3]

Players 1, 2 and 3 are brothers and they live together. Players 1 and 2 can make together a profit of one unit, that is, $v(1, 2) = 1$. Player 3 is a disabled person and can contribute nothing to any coalition. Therefore, $v\{1, 2, 3\} = 1$. Furthermore, we have $v\{1, 3\}$ =$v\{2, 3\} = 0$. Finally, we assume that $v\{i\} = 0 \; \forall i \in \{1, 2, 3\}$ for every player i. This is a classical unanimity game. The Shapley value of game is $Sh(N, v) = (\frac{1}{2}, \frac{1}{2}, 0)$. (Should the disabled brother leave his family?) If players 1 and 2 take responsibility for their brother (player 3), then the solidarity value $NR(N, v) = (\frac{7}{18}, \frac{7}{18}, \frac{4}{18})$.
But, following this rule, if $C_i^{av}(S) > C_i(v, S)$, then player i is *obliged* to offer some part of his/her marginal contribution to the coalition S to support some "weaker" members of S. With the Shapley value player 3 is a null player he gets 0, while with solidarity value player 3 is not a A-null player.

Example 2:

$N = \{1, 2, 3\}$

$v(1) = 1$,$v(1, 2) = 4$,$v(1, 2, 3) = 5$

$v(2) = 3$,$v(1, 3) = 1.5$

$v(3) = 0$, $v(2, 3) = 3$

We get the values $Sh(N, v) = (\frac{8.5}{6}, \frac{19}{6}, \frac{2.5}{6})$ and $NR(N, v) = (\frac{84}{54}, \frac{124.5}{54}, \frac{61.5}{54})$.

[3] Nowak and Radzik [1994]

Let compute the transfer from solidary players to player 3:

Example1/Players	Shapley value	Solidarity Value	Values Transferred to player 3
1	$\frac{1}{2}$	$\frac{7}{18}$	-0,11
2	$\frac{1}{2}$	$\frac{7}{18}$	-0,11
3	0	$\frac{4}{18}$	+0,222

Example2/Players	Shapley value	Solidarity Value	Values Transferred to player 3
1	$\frac{8{,}5}{6}$	$\frac{84}{54}$	+0,14
2	$\frac{19}{6}$	$\frac{124{,}5}{54}$	-0,86
3	$\frac{2{,}5}{6}$	$\frac{61{,}5}{54}$	+0,72

Remark

For example 1, players 1 and 2 transfer $22,22\%$ of their values to the null player 3. In example 2, player 3 is not a null player, he received $22,78\%$ and has a value near to player 1's value. Initially player 1 has three times the value of player 3. The solidarity value can lead to situations where the poor through transfers can achieve a more favorable status. As we see in the life of every day the solidarity is intended to keep people in a social level so that they can improve their standard of living by looking for better job opportunities, making training to adapt to the supply of labor market .... etc ...

3 The Free Solidarity Value

Let us now suppose that some players decide to be solidary and others not solidary. We have two types of players solidary and not solidary players. We note this coalition grouping of solidary players by S^*. Among these players, each will have $\frac{1}{|S \cap S^*|}$ of the sum of their marginal contributions[4] every time they find themselves in the same coalition S.

When the coalition is mixed, formed by both types of players, and to reflect certain social behavior of the players in a game we define the following two types of contributions:

$$\forall i \in \underset{S\cap S^* \neq \varnothing}{S \subseteq N \backslash \varnothing} : \begin{cases} C_i(v,S) = (V(S) - V(S\backslash i)) & if\, i \in (S\backslash(S \cap S^*)) \\ \tilde{C}_i^{av}(v,S) = \frac{1}{|S\cap S^*|}(\sum\limits_{k \in S \cap S^*} (V(S) - V(S\backslash k)) & if\, i \in (S \cap S^*) \end{cases}$$

Every non-solidary players, $i \in (S\backslash S \cap S^*)$, will have a value depending on its marginal contribution $C_i(v,S)$. And every solidary players, $i \in (S \cap S^*)$, will have a value according to its average marginal contribution $\tilde{C}_i^{av}(v,S)$.

[4]The solidary players can adopt another way of sharing the sum of their marginal contributions, other than the egalitarian sharing.

Note that, Contrary to the seminal paper of Nowak and Radzik (1994), the average marginal contribution $\tilde{C}_i^{av}(v,S)$ that we propose through the last formula may be partial; if the players of the same coalition are not all solidary.

The free solidarity value $\zeta_i(N,v,S^*)$ of a cooperative TU-game (N,v) , and containing S^* as exogenous coalition, is given by:

$$\zeta_i(N,v,S^*) = \begin{cases} \sum\limits_{S\ni i} \frac{(n-s)!(s-1)!}{n!}[v(S)-v(S\backslash\{i\}] = Sh_i(N,v) & if\, i \notin S^* \\ \sum\limits_{S\ni i} \frac{(n-s)!(s-1)!}{n!}[\frac{1}{|S\cap S^*|}(\sum\limits_{k\in S\cap S^*} (V(S)-V(S\backslash k))] & if\, i \in S^* \end{cases} \tag{3}$$

Remark

if $S^* = \varnothing$ or $S^* = \{i\}$, so $\zeta(N,v,S^*) = Sh(N,v)$.
if $S^* =\ N$,so $\zeta(N,v,S^*) = NR(N,v)$.

Let reconsider the example 1 with $S^* = \{1,3\}$, the first player is very attached to his third brother and decides to help him, while the second player is not affected by the disability of his brother

As $1,3 \in S^* = \{1,3\}$ then:

$\zeta_1(N,v,S^*) = \sum\limits_{1\in(T\cap\{1,3\})\neq\varnothing} \frac{(n-t)!(t-1)!}{n!}[\tilde{C}_1^{av}(v,T)]$

$= \frac{1}{3}.0 + \frac{1}{6}.1 + \frac{1}{6}.\frac{1}{2}.0 + \frac{1}{3}.\frac{1}{2}.1 = \frac{1}{6} + \frac{1}{6} = \frac{2}{6}$

$\zeta_3(N,v,S^*) = \sum\limits_{3\in(T\cap\{1,3\})\neq\varnothing} \frac{(n-t)!(t-1)!}{n!}[\tilde{C}_3^{av}(v,T)]$

$= \frac{1}{3}.0 + \frac{1}{6}.\frac{1}{2}.0 + \frac{1}{6}.0 + \frac{1}{3}.\frac{1}{2}.1 = \frac{1}{6} = \frac{1}{6}$

$2 \notin S^* = \{1,3\}$ we use the Shapley formula:

$\zeta_2(N,v,S^*) = \sum\limits_{T\ni 2} \frac{(n-t)!(t-1)!}{n!}[v(T)-v(T\backslash\{2\}] = Sh_2(N,v)$

$= \frac{1}{3}.0 + \frac{1}{6}.1 + \frac{1}{6}.0 + \frac{1}{3}.1 = \frac{1}{6} + \frac{1}{3} = \frac{3}{6}$

$\Longrightarrow \zeta(N,v,S^*) = (\frac{1}{3}, \frac{1}{2}, \frac{1}{6})$

For example 2, let assume that $S^* = \{1,3\}$, we use the following formula:

$\zeta_1(N,v,S^*) = \sum\limits_{1\in(S\cap\{1,3\})\neq\varnothing} \frac{(n-s)!(s-1)!}{n!}[\tilde{C}_1^{av}(v,S)]$

$= \frac{1}{3}.1 + \frac{1}{6}.1 + \frac{1}{6}.\frac{1}{2}.(0.5+1.5) + \frac{1}{3}.\frac{1}{2}.(2+1) = \frac{7}{6}$

$\zeta_3(N,v,S^*) = \sum\limits_{3\in(S\cap\{1,3\})\neq\varnothing} \frac{(n-s)!(s-1)!}{n!}[\tilde{C}_3^{av}(v,S)]$

$= \frac{1}{3}.0 + \frac{1}{6}.\frac{1}{2}.(0,5+1.5) + \frac{1}{6}.0 + \frac{1}{3}.\frac{1}{2}.(2+1) = \frac{4}{6}$

$2 \notin S^* = \{1,3\}$ we use the Shapley formula:
$\zeta_2(N,v,S^*) = \sum\limits_{S \ni 2} \frac{(n-s)!(s-1)!}{n!}[v(S) - v(S \backslash \{2\}] = Sh_2(N,v)$
$= \frac{1}{3}.3 + \frac{1}{6}.3 + \frac{1}{6}.3 + \frac{1}{3}.3.5 = \frac{19}{6}$
$\Longrightarrow \zeta(N,v,S^*) = (\frac{7}{6}, \frac{19}{6}, \frac{4}{6})$.
For example 2, let assume $S^* = \{1,2\}$ and compute the free solidarity values:

Example1/Players	Shapley value	Free Solidarity Value	Values Transferred to player 3
1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{6}$
2	$\frac{1}{2}$	$\frac{1}{2}$	0
3	0	$\frac{1}{6}$	$+\frac{1}{6}$

Example2/Players	Shapley value	Free Solidarity Value	Values Transferred to player 3
1	$\frac{8{,}5}{6}$	$\frac{7}{6}$	$\frac{1{,}5}{6}$
2	$\frac{19}{6}$	$\frac{19}{6}$	0
3	$\frac{2{,}5}{6}$	$\frac{4}{6}$	$+\frac{1{,}5}{6}$

Here the transfer from players 1 and 2 is of $16,6\%$ for example 1, and $2,14\%$ for example 2.

Let now consider the case with $S^* = \{2,3\}$ in example 2, we obtain

Example2/Players	Shapley value	Free Solidarity Value	Values Transferred to player 3
1	$\frac{8{,}5}{6}$	$\frac{8{,}5}{6}$	0
2	$\frac{19}{6}$	$\frac{15}{6}$	$-\frac{4}{6}$
3	$\frac{2{,}5}{6}$	$\frac{6{,}5}{6}$	$+\frac{4}{6}$

Here player 2 transfers $21,05\%$ of his value to player 3.

Remarks

1. According to the value of Shapley, the players behave in a individual way. Every player will have a value which depends only on its own marginal contribution in the coalitions which can form. they are not obliged to help their weaker partners.

2. With the solidarity value, all the players, who contribute to the coalitions more than the average marginal contribution, *should* support their weaker partners.

3. In our case, the not solidarity player, is not punished because he not agrees to help his brother (by consent) and therefore he is going to be able to master his act of solidarity to remain at least as satisfied as possible.

4 An Axiomatic Characterization

Let $\psi(N, v, S^*)$ a value for a cooperative TU-game $G(N, v, S^*)$. We now consider some axioms that are desirable for such a value:
Axiom A(1), efficiency (Eff): for any game $(N, v) \in G^N$,
$\sum_{i \in N} \psi_i(N, v, S^*) = v(N, S^*)$, i.e. $\psi(N, v, S^*)$ is a function value;
Axiom A(2), additivity (Add): for any games (N, v, S^*), $(N, w, S^*) \in G^N$, $\psi(N, v + w, S^*) = \psi(N, v, S^*) + \psi(N, w, S^*)$;
Axiom (A3), Unaffected Allocation of non-solidary players - (UA) : when players decide freely, to be solidary (by forming S^*) this should not affect neither the values of the other players who have chosen to stay out of solidarity, nor the value v. i.e., $\forall S \subseteq N, v(S, S^*) = v(S, \varnothing)$ and $\forall i \notin S^* \neq \varnothing$, $\psi_i(N, v, S^*) = \psi_i(N, v, \varnothing)$;
Axiom A(4), conditional symmetry (Cond Sym): if i, j are conditional symmetric, i.e., $\forall S \subseteq N \backslash \{i, j\}$, $v(S \cup \{i\}) = v(S \cup \{j\})$, then $\psi_i = \psi_j$ if $i, j \in S^*$ or $i, j \notin S^*$;
Axiom A(5): + **(Cond-null):** if player $i \in N$ is qualified by Cond-null player, in this case, according to its type, his contributions, without exception, are 0. i.e., in every coalition S containing player i :$\tilde{C}_i^{av}(v, S) = 0$ if $i \in S^*$ and $C_i(v, S) = 0$ if $i \notin S^*$.

Axioms A(1)-A(2) are standards.
The axiom A(3) is new, it means that the not solidary players, who are in some coalition S, behave as the entire coalition S is formed only by not solidary players. Their values in S depend on their marginal contributions and not on their average marginal contributions. For them, all the players in N, are considered as not solidarys.

The axiom A(4) is an adaptation of the standard SYM axiom for our case. Contrary to seminal works of Shapley (1953) and Nowak and Radzik (1994), we have two types of players; solidary and not solidary. One cannot compare two players of the different type according to their marginal contributions. It is for that we supposed a comparison according to their membership. If it happens that in every coalition T containing two agents of the same type with the same marginal contributions then, they should be rewarded equally.

The axiom A(5) can be seen as a combination of the axiom of the null player of Shapley (1953) and the axiom of the A-null player of Nowak and Radzik (1994). If it happens that in every coalition T containing some or all players of S^*, players in $\{S \cap S^*\}$ has a null marginal average contribution, then, according to A(5), $\forall i \in (S \cap S^*)$, $\psi_i(N, v, S^*) = 0$.
And, if it happens that every coalition T containing player $i \notin S^*$, the marginal contribution of player i is null, i.e., $C_i(v, S) = 0$, then according to A(5) this player gets nothing from the game.

We can now give our main result.

Theorem

A value $\psi : G^N \mapsto R^n$ satisfies efficiency, conditional symmetry, additivity, respect of freedom of choice and a conditional null player axiom if and only if $\psi = \zeta$. i.e., ψ is the free solidarity value.

To prove this result, we follow the literature and implement Shapley's proof. We first look to the unanimity game $w_{(T,S^*)}(.)$ defined in the following definition. We show that the values of unanimity game satisfying the axioms are uniquely defined. That each game v can be expressed as a linear combination of unanimity games, we deduce therefore that the value is unique. The last point of proof consist by showing that the analytical given value ζ satisfies the axioms.

let's begin with the following definition:

Definition : *(unanimity game)*

Let N a set of n agents and S^* the coalition of solidays players. For $T \subseteq N \backslash \varnothing$, the unanimity games $w_{(T,S^*)}(.)$ is such that $\forall S \subseteq N$,

$$w_{(T,S^*)}(S) = \begin{cases} \frac{|T \cap S^*|}{|T|} \begin{pmatrix} \mid S \cap S^* \mid \\ \\ \mid T \cap S^* \mid \end{pmatrix}^{-1} + \frac{|T \backslash T \cap S^*|}{|T|} & if\ S \supset T \\ 0 & otherwise \end{cases} \tag{4}$$

Where $\begin{pmatrix} \mid S \cap S^* \mid \\ \\ \mid T \cap S^* \mid \end{pmatrix}^{-1} = \frac{|T \cap S^*|!(|S \cap S^*| - |T \cap S^*|)!}{|S \cap S^*|!}$

We have the following lemma.

Lemma 1

$\forall T \subseteq N, T \neq \varnothing$ the game $w_{(T,S^*)}(S)$has the following properties :

1. If $S = T$, then $w_{(T,S^*)}(S) = 1$.
2. If $T \subset S$ and $S^* = \varnothing$, then $w_{(T,\varnothing)}(S) = 1$.
3. If $T \subset S$ with $(S \cap S^*) = (T \cap S^*) \cup E$, $(T \cap S^*) \cap E = \varnothing$, $(T \cap S^*) \neq \varnothing$ and $E \neq \varnothing$, then $w_{(T,S^*)}(S) = \frac{1}{|S \cap S^*|} \sum_{i \in S \cap S^*} w_{(T,S^*)}(S \backslash \{i\})$.

4. *The set* $\{w_{T,S^*}(S) : T \subseteq N\backslash\varnothing\}$ *is a linear basis for* G^N.

   The last point of lemma means that every TU game (N, v, S^*) can be expressed in the basis $(w_{T,S^*})_{T\subseteq N\backslash\varnothing}$ such that $\forall S \subseteq N$
   $v(S) = \sum_{T\subseteq N\backslash\varnothing} \alpha_T w_{T,S^*}(S)$.

Proof

1. When $S = T$ then,
   $w_{T,S^*}(T)= \frac{|T\cap S^*|}{|T|} \begin{pmatrix} \mid S\cap S^* \mid \\ \\ \mid T\cap S^* \mid \end{pmatrix}^{-1} + \frac{|T\backslash T\cap S^*|}{|T|}= (\frac{|T\cap S^*|}{|T|} \times 1) + \frac{|T\backslash T\cap S^*|}{|T|} = 1.$

2. If $T \subset S$ and $S^* = \varnothing$, then $w_{(T,\varnothing)}(S)= \frac{0}{|T|} \begin{pmatrix} 0 \\ \\ 0 \end{pmatrix}^{-1} + \frac{|T\backslash\varnothing|}{|T|} = \frac{|T|}{|T|} = 1.$

3. If $T \subset S$ with $(S \cap S^*) = (T \cap S^*) \cup E$, $(T \cap S^*) \cap E = \varnothing$, $(T \cap S^*) \neq \varnothing$ and $E \neq \varnothing$, then $w_{(T,S^*)}(S) = \frac{1}{|S\cap S^*|} \sum_{i\in S\cap S^*} w_{(T,S^*)}(S\backslash\{i\})$.
   Using the expression of $w_{(T,S^*)}(S)$ in (4), if $T \subset S$, we have:
   $w_{(T,S^*)}(S)-\frac{1}{|S\cap S^*|}\sum_{i\in S\cap S^*} w_{(T,S^*)}(S\backslash\{i\})= w_{(T,S^*)}(S)-\frac{1}{|S\cap S^*|}\sum_{i\in E} w_{(T,S^*)}(S\backslash\{i\})$
   $= \frac{|T\cap S^*|}{|T|} \begin{pmatrix} \mid (T\cap S^*)\cup E \mid \\ \\ \mid T\cap S^* \mid \end{pmatrix}^{-1} - \frac{|T\cap S^*|}{|T|} \frac{1}{|(T\cap S^*)\cup E|} \sum_{i\in E} \begin{pmatrix} \mid (T\cap S^*)\cup (E\backslash\{i\}) \mid \\ \\ \mid T\cap S^* \mid \end{pmatrix}^{-1}$
   $= \frac{|T\cap S^*|}{|T|} \frac{|T\cap S^*|!|(E)|!}{|(T\cap S^*)\cup E|!} - \frac{|T\cap S^*|}{|T|} \frac{1}{|(T\cap S^*)\cup E|} \mid E \mid \times (\frac{|T\cap S^*|!||E|-1|!}{||((T\cap S^*)\cup E)|-1|!})$
   $$= \frac{\mid T\cap S^* \mid}{\mid T \mid} \frac{\mid T\cap S^* \mid ! \mid (E) \mid !}{\mid (T\cap S^*)\cup E \mid !} - \frac{\mid T\cap S^* \mid}{\mid T \mid} \frac{\mid T\cap S^* \mid ! \mid (E) \mid !}{\mid (T\cap S^*)\cup E \mid !} = 0. \qquad (5)$$

4. *The set* $\{w_{T,S^*}(S) : T \subseteq N\backslash\varnothing\}$ *is a linear basis for* G^N.

Following Nowak and Radzik (1994), let $S_1, S_2, ..., S_k$ is a fixed sequence of possible coalitions S of N such that $k = 2^n - 1$ is a number of coalition. So, there are k *unanimity games.*

First, we classify the coalitions by ascending order of size as follows:
$1 =\mid S_1 \mid\leq\mid S_2 \mid\leq .... \leq\mid S_k \mid= n$.
We can write $\alpha_T w_{T,S^*}(S)$ for all $S \subseteq N$, in matrix form:

$S\backslash T$	S_1	S_2	$\cdots$	S_k
S_1	1	0	$\cdots$	0
S_2	$\alpha_{S_1} w_{(S_1,S^*)}(S_2)$	1	$\cdots$	0
$\vdots$	$\vdots$	$\vdots$	$\ddots$	0
S_k	$\alpha_{S_1} w_{(S_1,S^*)}(S_k)$	$\alpha_{S_2} w_{(S_2,S^*)}(S_k)$	$\cdots$	1

The matrix is a triangle matrix with non-null diagonal elements, then its determinant is non-null.This implies that *the set* $\{w_{T,S^*}(S) : T \subseteq N\backslash\varnothing\}$ *is a linear basis for* $G^{N.}$ $\square$

Proof of the main result

Let's show that the value function of a unanimity game is unique. Let $T \subseteq N\backslash\varnothing$ and $\alpha \in \mathbb{R}$. Let us prove that $\psi(N, \alpha w_{T,S^*}, S^*)$ is uniquely defined.

1. One must check that, whatever its type, any player who is not in T is a Cond-null player in the game $\alpha w_{T,S^*}(S)$, i.e., one must proofs that the marginal contribution ($i \notin S^*$) or the average marginal contribution ($i \in S^*$) of a player is equal to 0, thus by A4 one can say that $\forall i \notin T$, $\psi_i(N, \alpha w_{T,S^*}) = 0$.

   This is obvious when T is not a subset of S.
   When $T \subseteq S, i \notin T$, two cases are possible:
   - If $i \in S^*$, then $(S \cap S^*) = (T \cap S^*) \cup E\,, (T \cap S^*) \cap E = \varnothing,\ E \neq \varnothing$.
   Following (5),
   $\frac{1}{|S \cap S^*|} \sum\limits_{i \in (S \cap S^*)} [\alpha w_{(T,S^*)}(S) - \alpha w_{(T,S^*)}(S\backslash\{i\})] = \tilde{C}_i^{av}(\alpha w_{(T,S^*)}, S) = 0.$
   - If $i \notin S^*$.
   It's clear that $T \subset \{S\backslash\{i\}\}$ since $T \subset S, i \notin T$. Then $\alpha w_{(T,S^*)}(S) = \alpha w_{(T,S^*)}(S\backslash\{i\})$. So, $C_i(\alpha w_{(T,S^*)}, S) = 0$. Thus By A(5) we can say that, $\forall i \notin T$, $\psi_i(N, \alpha w_{(T,S^*)}) = 0$ i.e., it is a Cond-null player.

2. We must show that every two agents of the same type in T are conditional symmetric in the game $\alpha w_{T,S^*}(.)$. i.e.,we must show that if $(i,j) \in (T\backslash T \cap S^*)$ or $(i,j) \in (T \cap S^*)$ we have $C_i(\alpha w_{(T,S^*)}, S) = C_j(\alpha w_{(T,S^*)}, S)$. Thus, by A(4) we can say that $\psi_i(N, \alpha w_{(T,S^*)}) = \psi_j(N, \alpha w_{(T,S^*)})$ because they are of the same type.
   -If $(i,j) \in (T)^2$, then for all $S \supset T$:
   $\alpha w_{(T,S^*)}(S) - \alpha w_{(T,S^*)}(S\backslash\{i\}) = \alpha w_{(T,S^*)}(S) - \alpha w_{(T,S^*)}(S\backslash\{j\})$
   $= \frac{|T \cap S^*|}{|T|} \begin{pmatrix} \mid S \cap S^* \mid \\ \\ \mid T \cap S^* \mid \end{pmatrix}^{-1} + \frac{|T\backslash T \cap S^*|}{|T|} - 0.$
   By Cond Sym, if $(i,j) \in (T\backslash T \cap S^*)$ or $(i,j) \in (T \cap S^*)$ then, $\psi_i(N, \alpha w_{(T,S^*)}) = \psi_j(N, \alpha w_{(T,S^*)})$.

3. Since ψ is efficient, then;

$$\sum_{i\in N}\psi_i(N,\alpha w_{T,S^*},S^*)=\alpha w_{(T,S^*)}(N)=\alpha\left\{\frac{|T\cap S^*|}{|T|}\begin{pmatrix}\mid S^*\mid\\ \\ \mid T\cap S^*\mid\end{pmatrix}^{-1}+\frac{|T\backslash T\cap S^*|}{|T|}\right\}$$

$$=\sum_{i\in T}\psi_i(N,\alpha w_{(T,S^*)},S^*)+\sum_{i\notin T}\psi_i(N,\alpha w_{(T,S^*)},S^*)$$

$$=\sum_{i\in T\cap S^*}\psi_i(N,\alpha w_{(T,S^*)},S^*)+\sum_{i\in(T\backslash T\cap S^*)}\psi_i(N,\alpha w_{(T,S^*)},S^*)$$

(following A(3))

$$=\sum_{i\in T\cap S^*}\psi_i(N,\alpha w_{(T,S^*)},S^*)+\sum_{i\in(T\backslash T\cap S^*)}\psi_i(N,\alpha w_{(T,\varnothing)},\varnothing).$$ Then,

$$\sum_{i\in T\cap S^*}\psi_i(N,\alpha w_{(T,S^*)},S^*)=\alpha w_{(T,S^*)}(N)-\sum_{i\in(T\backslash T\cap S^*)}\psi_i(N,\alpha w_{(T,\varnothing)},\varnothing).$$

We remember that $\psi_i(N,\alpha w_{(T,\varnothing)},\varnothing)=\frac{\alpha}{|T|}$ because, when $S^*=\varnothing$:

$$\sum_{i\in N}\psi_i(N,\alpha w_{(T,\varnothing)},\varnothing)=\alpha w_{(T,\varnothing)}(N)=\sum_{i\in T}\psi_i(N,\alpha w_{(T,\varnothing)},\varnothing)+\sum_{i\notin T}\psi_i(N,\alpha w_{(T,\varnothing)},\varnothing)$$

$$=\sum_{i\in T}\psi_i(N,\alpha w_{(T,\varnothing)},\varnothing)+0$$ (following A(5))

then,

$\psi_i(N,\alpha w_{(T,\varnothing)},\varnothing)=\frac{\alpha w_{(T,\varnothing)}(N)}{|T|}$(following A(4))

$$=\alpha\frac{\left\{\frac{|0|}{|T|}\begin{pmatrix}\mid 0\mid\\ \\ \mid 0\mid\end{pmatrix}^{-1}+\frac{|T\backslash\varnothing|}{|T|}\right\}}{|T|}=\frac{\alpha}{|T|}$$

We return now to our equation,

$$\sum_{i\in T\cap S^*}\psi_i(N,\alpha w_{(T,S^*)},S^*)=\alpha w_{(T,S^*)}(N)-\sum_{i\in(T\backslash T\cap S^*)}\psi_i(N,\alpha w_{(T,\varnothing)},\varnothing)$$

$$=\alpha\left\{\frac{|T\cap S^*|}{|T|}\begin{pmatrix}\mid S^*\mid\\ \\ \mid T\cap S^*\mid\end{pmatrix}^{-1}+\frac{|T\backslash T\cap S^*|}{|T|}\right\}-\mid T\backslash T\cap S^*\mid(\frac{\alpha}{|T|})$$

$$=\alpha\left\{\frac{|T\cap S^*|}{|T|}\begin{pmatrix}\mid S^*\mid\\ \\ \mid T\cap S^*\mid\end{pmatrix}^{-1}\right\}.$$

Following A(4), any player $i\in T\cap S^*$ will have

$$\frac{\alpha\left\{\frac{|T\cap S^*|}{|T|}\begin{pmatrix}\mid S^*\mid\\ \\ \mid T\cap S^*\mid\end{pmatrix}^{-1}\right\}}{|T\cap S^*|}=\frac{\alpha\left\{\begin{pmatrix}\mid S^*\mid\\ \\ \mid T\cap S^*\mid\end{pmatrix}^{-1}\right\}}{|T|}$$

By A(3), any player $i \in T\backslash T \cap S^*$ will have $\frac{\alpha(\frac{|T\backslash T\cap S^*|}{|T|})}{|T\backslash T\cap S^*|} = \frac{\alpha}{|T|}$.

$$\psi_i(N, \alpha w_{(T,S^*)}, S^*) = \begin{cases} \frac{\alpha\left\{\begin{pmatrix} \mid S^* \mid \\ \\ \mid T \cap S^* \mid \end{pmatrix}^{-1}\right\}}{|T|} & i \in T \cap S^* \\ \frac{\alpha}{|T|} & i \in T\backslash T \cap S^* \\ 0 & i \notin T \end{cases}$$

This proves that $\psi(N, \alpha w_{T,S^*}, S^*)$ is uniquely defined.

And because of Add, there is a unique *value function* that satisfies the used axioms.
Let now $v \in G^N$. From lemma 1 (pt 4),$v(S) = \sum_{T \subseteq N\backslash\varnothing} \alpha_T w_{T,S^*}(S)$ then,

$\psi_i(v) = \psi_i(\sum_{T \subseteq N\backslash\varnothing} \alpha_T w_{T,S^*}(.))$.

From A (2)

$\psi_i(v) = \sum_{T \subseteq N\backslash\varnothing} \psi_i(N, \alpha_T w_{T,S^*}, S^*)$ then, $\psi_i(v)$ is unique.

□

We now turn to show the existence of the free solidarity value. Let (N, v, S^*) a TU game and we consider our proposed solution, defined by (3). We are going to check whether this allocation satisfies the axioms.

- Eff:

We now that Shapley value satisfies efficiency:

$$\sum_{i \in N} Sh_i(N, v) = 1$$

but

$$\sum_{i \in N} \xi_i(N, v, S^*) = \sum_{i \in S^*} \xi_i(N, v, S^*) + \sum_{i \notin S^*} \xi_i(N, v, S^*) =$$
$$\sum_{i \in S^*} \xi_i(N, v) + \sum_{i \notin S^*} Sh_i(N, v).$$

Therefore it suffice to prove that

$$\sum_{i \in S^*} \xi_i(N, v) = \sum_{i \in S^*} Sh_i(N, v)$$

Otherwise it suffice to prove that

$$0 = \sum_{i \in S^*} \sum_{S \ni i} \frac{(n-s)!(s-1)!}{n!} \left(v(S) - v(S\backslash\{i\}) - \frac{1}{\mid S \cap S^* \mid}\big(\sum_{k \in S \cap S^*} (V(S) - V(S\backslash k)) \right) =$$

$$\sum_{i\in S^*}\sum_{S\ni i}\frac{(n-s)!(s-1)!}{n!}\left(\frac{1}{\mid S\cap S^*\mid}(\sum_{k\in S\cap S^*}(V(S\backslash k)-V(S\backslash i))\right)=$$

$$\sum_{S\ni i}\frac{(n-s)!(s-1)!}{n!}\left(\frac{1}{\mid S\cap S^*\mid}\sum_{i\in S^*}(\sum_{k\in S\cap S^*}(V(S\backslash k)-V(S\backslash i))\right)=$$

$$\frac{(n-s)!(s-1)!}{n!}\left(\frac{1}{\mid S\cap S^*\mid}\sum_{i\in S\cap S^*}(\sum_{k\in S\cap S^*}(V(S\backslash k)-V(S\backslash i))\right)=$$

$$\frac{(n-s)!(s-1)!}{n!}\left(\frac{1}{\mid S\cap S^*\mid}\left(\mid S\cap S^*\mid\sum_{k\in S\cap S^*}V(S\backslash k)-\mid S\cap S^*\mid\sum_{i\in S\cap S^*}(V(S\backslash i))\right)\right)=0$$

Which proves that $\zeta(N,v,S^*)$ is an efficient value.

- Cond-null: Let i be a Cond-null player:

1. If $i\in S^*$, i.e., $\forall S\subseteq N,\ i\in(S\cap S^*)\neq\varnothing:\tilde{C}_i^{av}(v,S)=0$.
   So in this case, $\xi_i(N,v,S^*)=\sum\limits_{i\in(S\cap S^*)\neq\varnothing}\frac{(n-s)!(s-1)!}{n!}[\tilde{C}_i^{av}(v,S)]=0$.
2. - If $i\notin S^*$,i.e, $\forall S\subseteq N, i\in(S\backslash(S\cap S^*)),\ C_i(v,S)=0$
   So $\xi_i(N,v,S^*)=\sum\limits_{S\ni i}\frac{(n-s)!(s-1)!}{n!}[C_i(v,S)]=0$.
   Thus, free solidarity value satisfies Cond-null axiom, when $i\in S^*$or when $i\notin S^*$.

- Cond Sym: Let i and j be conditional symmetric players :

by A(4), for all $S\subseteq N\backslash\{i,j\}$ we have $v(S\cup\{i\})=v(S\cup\{j\})$.

1. if $i,j\notin S^*$:

$$\xi_i(N,v,S^*)=Sh_i(N,v)=\sum_{S\ni i}\frac{(n-s)!(s-1)!}{n!}[v(S)-v(S\backslash\{i\}]$$

Which can be written as:

$$Sh_i(N,v)=\sum_{i,j\notin S}\frac{(n-s-1)!(s)!}{n!}[v(S\cup i)-v(S)]+$$

$$\sum_{i,j\notin S}\frac{(n-s-2)!(s+1)!}{n!}[v(S\cup i\cup j)-v(S\cup j)]$$

As $v(S\cup i)=v(S\cup j)$ for each coalition S not containing i and j. Then $\xi_i(N,v,S^*)=\xi_j(N,v,S^*)$; Free solidarity value satisfies Cond Sym when $i,j\notin S^*$.

2. if $(i,j) \in S^*$ we have:

$$\xi_i(N,v,S^*) = \sum_{S \ni i} \frac{(n-s)!(s-1)!}{n!} \left(\frac{1}{\mid S \cap S^* \mid} \sum_{k \in S \cap S^*} [v(S) - v(S \backslash \{k\}] \right) =$$

$$\sum_{i \notin S} \frac{(n-s-1)!(s)!}{n!} \left(\frac{1}{\mid S \cap S^* \mid +1} \sum_{k \in S \cup i \cap S^*} [v(S \cup i) - v(S \cup i \backslash \{k\}] \right) =$$

$$\sum_{i \notin S} \frac{(n-s-1)!(s)!}{n!} \left(\frac{1}{\mid S \cap S^* \mid +1} \left(\sum_{k \in S \cap S^*, k \neq i} [v(S \cup i) - v(S \backslash \{k\} \cup i)] + v(S \cup i) - v(S) \right) \right)$$

Now as $v(S \cup i) = v(S \cup i)$ and $v(S \backslash \{k\} \cup i) = v(S \backslash \{k\} \cup j)$, then $\xi_i(N,v,S^*) = \xi_j(N,v,S^*)$. Free solidarity value satisfies Cond Sym when $(i,j) \in S^{*2}$.

- Unaffected Allocation of non-solidary players - **(UA)**:

let a game $(N,v,S^*) \in G^N$, for all $i \notin S^*$, we have ;

$\psi_i(N,v,S^*) = \sum\limits_{i \in S \subseteq N} \frac{(n-s)!(s-1)!}{n!} [v(S,S^*) - v(S \backslash \{i\}, S^*)]$

$= \sum\limits_{i \in S \subseteq N} \frac{(n-s)!(s-1)!}{n!} [v(S,\varnothing) - v(S \backslash \{i\}), \varnothing)] = \psi_i(N,v,\varnothing)$

- Add : ξ is linear, so Add is satisfied.

□

Discussion and concluding remarks

According to the value of Shapley, the players behave in a individual way. Every player will have a value which depends only on its own marginal contribution in the coalitions which may form. The value of agent i in coalition S is the average marginal value over all possible orders in which the agents may join the coalition.

For Nowak and Radzik (1994), all the players behave in a solidarity way. The players, who contribute to the coalitions more than the average marginal contribution, *should* support their weaker partners. Formally the solidarity value for a player i appears as a weighted sum of the average contribution of all coalitions of player in the game.

As here conceived, the principle of solidarity in this work is free: each player can give a portion of his payments and so can promote (increase satisfaction of) one or more players without disadvantaging other players in the game.

We showed that the free solidarity value can improve the solution of Shapley and the one of Nowak and Radzik. The solidary players are not punished, because they agree to help their weaker partners (by consent) and therefore they are going to be able to master their act of solidarity to remain at least as satisfied as possible.

Solidarity is thus far from being mandatory: rather, it is a natural act and decided before the game. To present this, we assumed that an exogenous coalition is freely formed before the game begins and includes solidary players.

Thus, throughout this paper, it has been shown that taking into account the possible existence of an exogenous coalition of solidary players, changes and somewhat improves the issue of sharing. Calculation formulas were presented, based on both the solidarity value of Nowak and Radzik and the Shapley value. In order to take into account this type of coalition, we have supplied, using an axiomatic approach, a new interpretation of Shapley-Solidarity technique for sharing, named *free solidarity value.*

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