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Scoring Rules over Subsets of Alternatives: Consistency and Paradoxes

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Scoring Rules over Subsets of Alternatives : Consistency and Paradoxes*

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Abstract

We know since the works of Gehrlein and Fishburn (1980, 1981), Fishburn (1981) and Saari (1987, 1988, 1990) that, the collective rankings of scoring rules are not stable when some alternatives are dropped from the set of alternatives. However, in the literature, attention has been mainly devoted to the relationship between pairwise majority vote and scoring rules rankings. In this paper, we focus on the relationships between four-candidate and three-candidate rankings. More precisely, given a collective ranking over a set of four candidates, we determine under the impartial culture condition, the probability of each of the six possible rankings to occur when one candidate is dropped. As a consequence, we derive from our computations, the likelihood of two paradoxes of committee elections, the Leaving Member paradox (Staring, 1986) and of the Prior Successor Paradox which occur when an elected candidate steps down from a two-member committee.

JEL Classification: D70, D71

Keywords: Scoring rule, Consistency, Collective Ranking, Committee, Paradox, Impartial Culture.

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1 Introduction

One of the main objectives of the social choice theory is the aggregation of individual preferences into a collective ranking that is the determination of a complete order over the set of the alternatives or candidates. This objective can be achieved when every voter gives points to each candidate in accordance with his preference order. Hence, the candidate with the highest score will be ranked at the top and the one with the lowest score will be at the bottom of the collective ranking. The Plurality rule, the Antiplurality rule and the Borda rule are among others, some well known scoring rules that can be used for a such objective. With the Plurality rule, a candidate's score is the number of times he is top ranked in the individual rankings. With the Antiplurality rule, a candidate's score is equal to the number of voters that do not rank him at the bottom of their rankings. With m candidates, the Borda rule gives $m - j$ points to a candidate each time, he is ranked j-th in a voter's ranking. The total number of points received by a candidate, defines his Borda's score.

Some questions that have been well addressed in the literature are the following : given a set of candidates, how a collective ranking can be altered after one or more candidates are removed from the competition? Is the new collective ranking consistent with the former one? These questions can be tracked back to the Borda-Condorcet debate. At the end of the 18th century, Borda (1781) and Condorcet (1785) who were members of the Paris Royal Academy of Sciences, tried to propose an alternative voting rule to the one that was in use in the academy (see McLean and Urken (1995)). The *Borda rule* picks as the winner, the candidate with the highest *Borda's score*. Condorcet (1785) criticized the *Borda* rule in that it can exist a candidate that is preferred by more than half of the electorate to the *Borda winner*. Condorcet (1785) proposed a rule based on pairwise comparisons[1]. According to this rule, a candidate should be declared the winner if he beats all the other candidates in pairwise majority; such a candidate is called the *Condorcet winner*. Nonetheless, the *Condorcet rule* has a main drawback : it can lead to *cycles*[2] in some circumstances. Though the Borda-Condorcet debate[3] about the choice of the best voting rule is still alive, everyone agrees that they were the first authors who emphasized the fact that *scoring rules* and *pairwise comparisons* are not always consistent.

The Borda-Condorcet debate can be envisaged in a broader picture : what are

[1] See Young (1988) for a modern interpretation of Condorcet's rule.

[2] Let us take three candidates a, b and c in order to illustrate what is a cycle in a simple way. For a given electorate, if a is majority preferred to b and b is majority preferred to c and c is majority preferred to a, this describe a voting cycle with three candidates.

[3] Finally, the Borda rule was retained by the Academy.

the relationships between the rankings on a set A of candidates and the rankings on the subset B included in A for a given preference profile? Apart from classical studies analyzing the relationships between pairwise voting and scoring rules (see Dodgson (1876); Nanson (1882); Smith (1973); Fishburn and Gehrlein (1976)), the first extension of these is due to Fishburn (1981) who showed that there always exist a preference profile for which removing any candidate from A leads to the reversed ranking on the remaining candidates. In a seminal paper Saari (1988) generalized this result by studying simultaneously the ranking on all the subsets of A for a given profile. He showed that, for most of the scoring rules (the Borda rule being one of the few exceptions) anything can happen for some profiles and no relationship prevails. This result was further developed in Saari (1987, 1988, 1990, 1996).

This historical result could cast a doubt on the practical use of scoring rules. What remains is to see whether this paradoxical results are just rare oddities or betray more generalized behavior. In modern social choice theory, many works have tried to analyze the relationships between pairwise and scoring rules. Among others, we can mention the works of Gehrlein and Fishburn (1976), Gehrlein and Fishburn (1980, 1981), Gehrlein et al. (1982), Fishburn (1981), Tataru and Merlin (1997), Van Newenhizen (1992) and more recently, Cervone *et al.* (2005). Most of them prove that given a scoring rule and a collective ranking, there is no reason to think that the pairwise comparisons will always be consistent with this collective ranking.

However, this line of research barely analyzed anything but pairwise relationships. In three-candidate elections, Gehrlein and Fishburn (1980, 1981) computed the limit probabilities under the Impartial Culture (IC) condition (defined later) that, given a scoring rule, the pairwise comparisons between candidates agree or are consistent with the collective ranking. They showed that, the agreement is maximized by the Borda rule and is minimized[4] by the Plurality rule and the Antiplurality rule.

Only one reference dealt with the relationship between the four-candidate set and the three-candidate subsets: Gehrlein and Fishburn (1980). They computed the likelihood that a ranking on a three-candidate subset is lifted up to the four-candidate subset under the IC condition. They concluded that, the probability of agreement between pairwise comparisons and the collective ranking is maximized by the Borda rule. In the same paper, they computed the mean limit probability that the collective ranking on three candidates agree with the collective ranking on four

[4] Gehrlein and Fishburn (1980, 1981) showed that with three candidates a, b, c, given that the collective ranking is abc, the limit probability to have a majority preferred to b (or b majority preferred to c) is 85.3% for the Borda rule and 75.5% for the Plurality rule and the Antiplurality rule; the limit probability to have a majority preferred to c is 96.9% for the Borda rule and 90.1% for the Plurality rule and the Antiplurality rule.

candidates. One drawback of their approach is that, they evaluated the likelihood of the same ranking on $\{a, b, c, d\}$ and $\{a, b, c\}$ regardless of the position of d in the four-candidate ranking; they only cared about the possibility of lifting up the initial ranking on $\{a, b, c\}$ to the superset.

Our objective in this paper will be to derive the probability of any ranking on the three-candidate subset given that any candidate has been removed from the four-candidate set[5]. In this paper, we enrich Gehrlein and Fishburn (1980)'s analysis 1) *by obtaining exact probabilities of consistency depending on the original position of the removed candidate* 2) *by deriving in each case, the likelihood of all the possible rankings on subsets*. Our probability computations not only lead us to make an hierarchy of the main scoring rules according to their stability; we also derive from them, the likelihood of some electoral paradoxes for committee elections.

Assume that, when electing a committee of size g, this committee is made by the candidates with the g greatest scores (the g top ranked candidates of the collective ranking). The first paradox we deal with is the *Prior Successor Paradox* (PSP). Since the elected committee is formed by the candidates with the g greatest scores, we define the Prior Successor as the candidate with the *g+1-th* best score. The PSP occurs if after a member of the elected committee leaves, a new ballot (given the subset of candidates) leads *ceteris paribus*[6] to a committee containing all the $g - 1$ members of the previous committee without the Prior Successor. The other paradox we define and deal with is more severe than the PSP : the *Leaving Member Paradox* (LMP) due to Staring (1986); it occurs when after a member of an elected committee leaves, a new ballot, *ceteris paribus*, leads to a new committee without some members of the previous ones; even worse, the two committees may be disjoint.

The rest of the paper is structured as follows : Section 2 is devoted to basic notations and definitions. In Section 3, we motivate the paper by considering some examples showing how the collective rankings over proper subsets of a set of alternatives can be consistent or not with the collective ranking of this set. In Section 4, we evaluate this event in the four-alternatives case using the impartial culture condition. In Section 5, we provide the formal definitions of the PSP and the LMP and we then derive their likelihood in four-candidate elections and two-member committees

[5]Notice that, in Gehrlein and Fishburn (1980, 1981), Fishburn (1981) and Saari (1987, 1988, 1990, 1996) as it will the case in this paper, when a candidate is removed, it will not be for strategic or for manipulation purpose as in Tideman (1987) or Dutta *et al.* (2001). Removing a candidate is strategic if the objective is to improve the rank of a certain candidate. Results on this issue have been recently discovered by Lang *et al.* (2013). We are not going to say more about this point since we are not concerned with the strategic aspect of the withdrawing of one or more candidates.

[6]This means that, voters keep their preferences unchanged on the rest of candidates no matter who is the leaving candidate.

as a consequence of our probability computations. Section 6 concludes. Details on probability computations are provided in Appendices A and B (Section 7).

2 Notation and definitions

2.1 Preferences

Let N be the set of n voters ($n \geq 2$) and A the set of m alternatives or candidates, $m \geq 3$. The binary relation R over A is a subset of the cartesian product $A \times A$. For $a, b \in A$, if $(a, b) \in R$, we write aRb to say that "a is at least good as b". $\neg aRb$ is the negation of aRb. If we have aRb and $\neg bRa$, we will say that "a is better or strictly preferred to b". In this case, we write aPb with P the asymmetric component of R. The symmetric component of R, I, is defined by aIb denoting an indifference between a and b *i.e* aRb and bRa. The preference profile $\pi = (P_1, P_2, ..., P_i, ..., P_n)$ gives all the linear orders[7] of all the n voters on A where P_i is the strict ranking of a given voter i. When we consider the preference of voter i on $B \subset A$, we will simply use the restriction of P_i to B. The set of all the preference profiles of size n on A is denoted by $\mathcal{P}(A)^n$. A voting situation $\tilde{n} = (n_1, n_2, ..., n_t, ..., n_{m!})$ indicates the number of voters for each linear order such that $\sum_{t=1}^{m!} n_t = n$. In the subsequent, we simply write $\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$ or $abcd$ to say that a is strictly preferred to b, b strictly preferred to c and c strictly preferred to d. Table 2.1 gives the labels of all the 24 types of strict rankings with four candidates.

Table 2.1: Labels of preferences on $A = \{a, b, c, d\}$

n_1	n_2	n_3	n_4	n_5	n_6	n_7	n_8	n_9	n_{10}	n_{11}	n_{12}
a	a	a	a	a	a	b	b	b	b	b	b
b	b	c	c	d	d	a	a	c	c	d	d
c	d	b	d	b	c	c	d	a	d	a	c
d	c	d	b	c	b	d	c	d	a	c	a

n_{13}	n_{14}	n_{15}	n_{16}	n_{17}	n_{18}	n_{19}	n_{20}	n_{21}	n_{22}	n_{23}	n_{24}
c	c	c	c	c	c	d	d	d	d	d	d
a	a	b	b	d	d	a	a	b	b	c	c
b	d	a	d	a	b	b	c	a	c	a	b
d	b	d	a	b	a	c	b	c	a	b	a

[7]A linear order is a binary relation that is transitive, complete and antisymmetric. The binary relation R on A is *transitive* if for $a, b, c \in A$, if aRb and bRc then aRc. R is *antisymmetric* if for all for $a \neq b$, $aRb \Rightarrow \neg bRa$; if we have aRb and bRa, then $a = b$. R is *complete* if and only if for all $a, b \in A$, we have aRb or bRa.

When the number of voters who rank a before b is greater than that of those who b before a, a is said majority preferred to b; we denote it by $aM(\pi)b$ or simply aMb.

2.2 Scoring rules

A scoring rule is a voting system that gives points to candidates in accordance with the position they occupy in voters rankings. The total number of points received by a candidate defines his scores for the considered rule. The winner is the one with the greatest score.

Let $r(i, a, A)$ for short when the context is clear, be the rank of candidate $a \in A$ in voter i's ranking.

$$r(i, a, A) = \sharp\{z \in B : zP_ia\} + 1$$

In a similar way we defines $r(i, a, B)$ for $a \in B \subseteq A$.

We denote by $w = (w_1, w_2, w_3, ..., w_j, ..., w_m)$ the scoring vector associated to the voting system such that $w_1 = 1 \geq w_2 \geq ... \geq w_j \geq ... \geq w_m = 0$. The most famous scoring rules are:

- The vector w such that $w_2 = ... = w_m = 0$ defines the *simple Plurality* rule: voters only cast a vote for their top ranked candidate. For $m = 4$, the scoring vector of the *simple Plurality* rule is $w_P = (1, 0, 0, 0)$.

- For $w_2 = ... = w_{m-1} = 1$, we have the *Antiplurality* rule under which each voter vote for all the candidates except his bottom ranked candidate; with $m = 4$, the scoring vector of the *Antiplurality* rule is $w_{Ap} = (1, 1, 1, 0)$.

- With m candidates, the *Borda* rule gives $\frac{m-j}{m-1}$ points to a candidate each time he is ranked *j-th*; then, the associated scoring vector is $w = (1, \frac{m-2}{m-1}, ..., \frac{m-j}{m-1}, ..., \frac{1}{m-1}, 0)$. In the four-candidate case, the tally of the Borda rule is $w_{Bor} = (1, \frac{2}{3}, \frac{1}{3}, 0)$.

- Under the *Limited Voting*, voters vote for exactly g candidates $(0 < g < m)$[8]. For $g = 1$, the Limited Voting is equivalent to the Simple Plurality rule and to the Antiplurality for $g = m - 1$. In this paper, as $m = 4$, we will take $g = 2$. So, the associated tally is $w_{LV} = (1, 1, 0, 0)$.

- The two others rules we are going to deal with are the *Plurality extension* and the *Antiplurality extension* both defined by Saari (1996). We will say more on these rules in Section 3. For $m = 4$, the tally of the Plurality extension is $w_{Pe} = (1, \frac{1}{3}, 0, 0)$ and that of the Antiplurality extension rule is $w_{Ape} = (1, 1, \frac{2}{3}, 0)$.

[8]For an overview on this voting system and those related to it, see Dummet (1984).

Given the voting system w, for a candidate $a \in A$, his score is given by :

$$S(A, w, \pi, a) = \sum_{i=1}^{n} w_{r(i,a,A)}$$

We denote by $R(A, w, \pi)$ ($R(A, \pi)$ for short) the collective ranking on A given the profile π for the scoring vector w. For all $B \subseteq A$ with $|B| \geq 2$, we define w^B the scoring vector of dimension $|B|$ we use to rank alternatives of the subset B. As with w, we have $w^B = (w_1^B, w_2^B, ..., w_{\sharp B}^B)$ such that $w_1^B = 1 \geq w_2^B \geq ... \geq w_{\sharp B}^B = 0$. So, for a given subset B, the preference P_i of voter i is now defined by its restriction to B. Hence, given w^B, for a candidate $a \in B$, his score is given by :

$$S(B, w, \pi, a) = \sum_{i=1}^{n} w_{r(i,a,B)}^B$$

We denote by $R(\pi, B)$ the collective ranking on B when the profile π is restricted to B. We can now define a generalized scoring rule

$$\omega = (w^{B^1}, w^{B^2}, \ldots, w^{B^j}, \ldots, w^{B^{2^m-(m-1)}})$$

as the collection of scoring vectors we use for each proper subset B of A.

In this paper, we will focus on $|A| = 4$ and $|B| = 3$. So, if $A = \{a, b, c, d\}$ then $B \in \Big\{\{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}\Big\}$. Assuming that the same scoring rule is used on both the whole set and on the three candidate subsets, Table 2.2 gives for each of the voting rules analyzed in this paper, the associated tallies.

Table 2.2: Voting rules and tallies

	Tallies	
Voting rules	w	w^B
Plurality	$(1, 0, 0, 0)$	$(1, 0, 0)$
Saari's Plurality extension	$(1, \frac{1}{3}, 0, 0)$	$(1, 0, 0)$
Antiplurality	$(1, 1, 1, 0)$	$(1, 1, 0)$
Saari's Antiplurality extension	$(1, 1, \frac{2}{3}, 0)$	$(1, 1, 0)$
Borda	$(1, \frac{2}{3}, \frac{1}{3}, 0)$	$(1, \frac{1}{2}, 0)$
Limited Voting	$(1, 1, 0, 0)$	$(1, 1, 0)$

3 The dictionary

When restricting the profile π from A to B the following events can occur :

- $R(\pi, B) = R(\pi, A)$. The candidates in the collective ranking on B appear in the same order as in the collective ranking on A. In a such case, we will say that *the scoring rule w^B agree or is consistent with the scoring rule w.*

- $R(\pi, B) \neq R(\pi, A)$. The candidates in the collective ranking on B dot not appear in the same order as in the collective ranking on A. In a such case, we will say that *the scoring rule w^B is not consistent with the scoring rule w.* Even worse, the candidates in the collective ranking on B may appear in the reversed order of that of the collective ranking on A, as shown in Example 1.

Example 1. *For $A = \{a, b, c, d\}$, consider the following preference profile with 102 voters*[9].

12	6	6	3	6	12	6	7	6	12	14	6	6
a	a	a	a	b	b	b	c	c	c	d	d	d
b	b	d	d	c	c	d	a	a	d	a	b	c
c	d	b	c	a	d	c	b	d	a	b	c	a
d	c	c	b	d	a	a	d	b	b	c	a	b

For a given scoring vector $w = (1, w_2, w_3, 0)$, we get,

$$S(A, w, \pi, a) = 27 + 27w_2 + 24w_3 \qquad S(A, w, \pi, b) = 24 + 24w_2 + 27w_3$$

$$S(A, w, \pi, c) = 25 + 24w_2 + 27w_3 \qquad S(A, w, \pi, d) = 26 + 27w_2 + 24w_3$$

Since $w_2 \geq w_3$, it can be easily checked that $R(\pi, A) = adcb$.

Suppose that b is dropped. Using the scoring vector $w^B = (1, w_2^B, 0)$ on $B = \{a, c, d\}$, we have

$$S(B, w^B, \pi, a) = 27+33w_2^B, \quad S(B, w^B, \pi, c) = 43+30w_2^B, \quad S(B, w^B, \pi, c) = 33+39w_2^B$$

Since $R(\pi, B) = cda$ for $w^B = 0$ and $w^B = 1$, it comes that $R(\pi, B) = cda$ for all w^B.

So, with this preference profile, for every couple (w, w^B) of scoring vectors, given that $R(\pi, A) = adcb$, if b is removed, this lead to a complete reversal ranking on the remaining candidates

[9] As we take 102 voters, this does not mean that with less than 102 voters there is no the inconsistency. The ready is free to build example for his own.

All in all, what kind of consistency and inconsistency can appear for a given generalized scoring rule ω? Saari studied extensively this issue in a series of seminal papers (Saari (1987, 1988, 1990, 1996)).

Consider $\mathcal{P}(A)^\infty = \cup_{n=1}^{\infty} \mathcal{P}(A)^n$, the set of the possible preference profiles over A, for any population size. Let $\kappa = 2^m - (m-1)$. For $B \subseteq A$, a voting rule f^B associates to each profile in $\mathcal{P}(A)^\infty$ a collective ranking in $\mathcal{R}(B)$ the set of all the collective rankings on B. A generalized voting rule Φ^m is a list of voting rules, one for each subset:

$$\Phi^m = (f^{B^1}, f^{B^2}, \dots f^{B^\kappa}).$$

Hence, for each profile $\pi \in \mathcal{P}(A)^\infty$, we obtain κ rankings, one for each subset:

$$\begin{array}{rcl} \Phi^m : \mathcal{P}(A)^\infty & \to & \mathcal{R}(B^1) \times \mathcal{R}(B^2) \times \dots \mathcal{R}(B^\kappa) = \mathcal{U}^m \\ \pi & \to & (R(\pi, B^1), R(\pi, B^2), \dots, R(\pi, B^m)) = \mu \end{array}$$

In Saari's terminology, the result of a generalized voting rule for a given profile π and for each subset is called a *word* μ, and $\mathcal{U}^m$ is the universal domain, the set of all possible entries. The set of all achievable words for a given Φ^m as we enumerate the set of possible preferences is then called the *Dictionary* of the voting rule:

$$\mathcal{D}(\Phi^m) = \{\mu \in \mathcal{U}^m : \Phi^m(\pi) = \mu,\, \pi \in P(A)^\infty\}$$

Let us denote by Bor^m the generalized scoring rule that uses the Borda count on each subset. Similarly, we define the generalized version of the simple Plurality rule (P^m) and the Antiplurality rule (Ap^m). More generally, F^m_ω is the generalized scoring rule using the family of scoring vectors in ω. The first Theorem establishes the superiority of the Borda count among the class of generalized scoring rules.

Theorem 1. *(Saari, 1988) Consider a generalized scoring rule F^m_ω, different from the Borda count. Hence:*

- *For $m = 3$, $\mathcal{D}(Bor^m) \subset \mathcal{D}(F^m_\omega) = \mathcal{U}^m$.*
- *For $m \geq 4$, $\mathcal{D}(Bor^m) \subset \mathcal{D}(F^m_\omega) \subseteq \mathcal{D}(P^m) = \mathcal{D}(Ap^m) = \mathcal{U}^m$.*

The interpretation of the result is clear: if an inconsistency occurs for some profile with the Borda count, it will also occur (for the same profile or a different one) for any other generalized scoring rule. Moreover, the Borda count uniquely minimizes the number of possible words in this class of voting rules[10]. Though Borda can be

[10]However, this is not the case if we seek for generalized voting rules in other families. By nature, dictatorship is perfectly consistent. More interestingly, Saari and Merlin (1996) proved that for the *Copeland rule* Cop^m, which ranks the candidates on their number of pairwise victories, $\mathcal{D}(Cop^m) \subset \mathcal{D}(Bor^m)$.

viewed as the optimal scoring rule, it is not free from severe paradoxes such as the reversal paradox displayed in Example 1. What kind of relationships can we expect when we use the Borda count. The answer is the generalization of a well know results in social choice literature : The Condorcet winner (*i.e* the candidate who is able to defeat any other opponent in pairwise comparison) is never ranked last by the Borda count and the Condorcet loser (*i.e* the candidate who is defeated by any other alternative) is never ranked first by the Borda count[11].

Theorem 2. *(Saari, 1990) Assume that candidate a is first (resp. last) ranked in all the subsets B of size k. Hence, he cannot be ranked last (resp. first) in the supersets of size $l \geq k$.*

Furthermore, he describes in the four candidate case, for which voting rules similar conclusion can be reached.

Theorem 3. *Saari (1996) Consider the scoring vector $w = (w_1, w_2, w_3, w_4)$ to be used in A and the scoring vector $w^B = (w_1^B, w_2^B, w_3^B)$ to be used in all the subsets B of three candidates. The same relationships as in Theorem 2 exist between the three candidate and the four candidate rankings if and only if:*

$$w_1 = 3w_1^B,\ w_2 = w_1^B + 2w_2^B,\ w_3 = 2w_2^B$$

First, it is immediate that the natural extension of $w^B = (1, \frac{1}{2}, 0)$ is $w = (3, 2, 1, 0)$, that is, the non normalized version of the Borda count. Hence, Borda maps into Borda. It is then easy to check that the natural extension of the plurality rule $w^B = (1, 0, 0)$ is the vector $w = (3, 1, 0, 0)$ or $w_{Pe} = (1, \frac{1}{3}, 0, 0)$ in its normalized version. Similarly, $w^B = (1, 1, 0)$ maps into $w_{Ape} = (1, 1, \frac{2}{3}, 0)$. One can also notice that it is impossible to find a solution to the system:

$$\begin{cases} 1 = 3w_1^B \\ 1 = w_1^B + 2w_2^B \\ 0 = 2w_2^B \end{cases}$$

Hence, no relationship at all exist between w_{LV} and $w^B = (w_1^B, w_2^B, w_3^B)$.

Theorems 1 to 3 establish the superiority of the Borda count in the class of scoring rule when one wishes to minimizes the types of inconsistencies we can observe across subsets. However, they do not tell us whether the likelihood of paradoxes are rare oddities or not, and whether the Borda count also minimizes the probability of

[11]This results can be tracked back to Nanson (1882). Modern proofs are proposed by Smith (1973) and Fishburn and Gehrlein (1976).

the different paradoxes. The vast literature on the relationships between pairwise majority and scoring rules gives a positive answer to this last issue (See Gehrlein and Lepelley (2010) for a survey). For almost all the a priori probability distribution one can imagine on the set of preferences profiles[12] the Borda count is the most likely to lift up the results on pairwise comparisons to supersets[13].

By estimating the likelihood of consistent and inconsistent rankings on four candidate and three candidate subsets, the current paper wishes to examine wether the Borda count will still prevails. Theorem 3 also suggest that new rules, like the plurality extension and the antiplurality extensions, could fare well. In order to answer all these questions, we need to set an a priori assumption on the likelihood of the different profile. In a first step, we chose the most common one, namely, the Impartial Culture assumption.

4 Probabilities of consistency with four alternatives

4.1 A probabilistic model: The impartial culture

One of the most used assumptions in the social choice literature when computing the likelihood of given events is the *Impartial Culture* (IC). Under IC, it is assumed that, each voter chooses her preference following a uniform probability distribution. It gives probability $\frac{1}{m!}$ to each ranking of being chosen independently. The likelihood of a given voting situation $\tilde{n} = (n_1, n_2, ..., n_t, ..., n_{m!})$ is

$$Prob(\tilde{n} = (n_1, n_2, ..., n_t, ..., n_{m!})) = \frac{n!}{\prod_{i=1}^{m!} n_i!} \times (m!)^{-n}$$

For more details about the IC and other probabilistic assumptions, see among others Gehrlein and Fishburn (1976), Berg and Lepelley (1994), Gehrlein and Lepelley (2010).

[12] We here only consider the probability models that are neutral, *i.e*, that treat equally all alternatives and all the rankings

[13] A few exception has been reported concerning the selection of the Condorcet winner in the three candidate elections. Under an assumption called the Impartial Anonymous Culture, Cervone *et al.* (2005) showed that though the Borda count is close to be optimal, the real optimum is obtained with $v = (1, 0.37225, 0)$ as n goes to infinity. The optimality of the Borda count can also be contested by Approval Voting in some specific scenarios when one considers indifferent voters, as in Diss *et al.* (2010).

4.2 The consistency probabilities : Gehrlein and Fishburn's results

Using the IC assumption, Gehrlein and Fishburn (1980) have computed in four-candidate elections *1)*-the limit probability that the pairwise comparisons are consistent with the collective ranking and *2)*-the mean limit probability that the collective ranking on three candidates agree with the collective ranking on four candidates.

Consider $A = \{a, b, c, d\}$ a set of four alternatives and B a proper subset of A with two alternatives such that $B \in \{\{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}\}$. Let us denote by $q((w, w^B), xy)$ $(x, y \in A, x \neq y)$ the limit probability under IC that the pairwise comparison between x and y is consistent with how x and y appear in the collective ranking on A. So, if $abcd$ is the collective ranking, $q((w, w^B), ab)$ will be the probability that a is majority preferred to b or the probability that a still ranked before b when the two other candidates are removed. Gehrlein and Fishburn (1980) provided the expression of each of the $q((w, w^B), xy)$. We are not going to recall these expressions here. They showed that

$$q((w, w^B), ab) = q((w, w^B), cd) \quad \text{and} \quad q((w, w^B), ac) = q((w, w^B), bd) \tag{4.1}$$

Also, they showed that each $q((w, w^B), xy)$ is maximized by the Borda rule and minimized by the Plurality rule and the Antiplurality rule. In Table 4.1, we report their results for the Borda rule, the Plurality rule and the Antiplurality rule. Also, we use their expressions of $q((w, w^B), xy)$ to derive the results for the Limited Voting, the Plurality extension and the Antiplurality extension.

Table 4.1: Scoring rules and consistency probabilities over two-alternatives subsets

	$q((w, w^B), ab)$	$q((w, w^B), ac)$	$q((w, w^B), ad)$	$q((w, w^B), bc)$
Borda	0.799847	0.924842	0.980402	0.767058
Plurality Antiplurality	0.680513	0.796887	0.893273	0.651925
Limited Voting	0.731399	0.859501	0.943660	0.699058
Plurality extended Antiplurality extended	0.741092	0.870079	0.950731	0.708349

Apart from confirming the optimality of the Borda rule, it comes from Table 4.1 that, the extended rules that perform better than the Limited voting which is in turn, better than the Plurality rule and the Antiplurality rule.

In the four-candidate case, Gehrlein and Fishburn (1980) computed[14] the limit probability given the collective ranking on $A = \{a, b, c, d\}$ that the new collective ranking between a, b and c is consistent with the former one after alternative d is removed no matter his position in that ranking[15]. They showed that this probability is maximized by the Borda rule and minimized by the two Plurality-Antiplurality score vectors combination $i.e$ $w = (1, 0, 0, 0)$ and $w^B = (1, 1, 0)$.

In the four-candidate case, Gehrlein and Fishburn (1980)'s results are somewhat limited in scope since they leave some questions unanswered. Is the agreement probability the same when the removed candidate was first, second, third or last in the original ranking? When a four-alternative set is restricted to a three-alternative proper subset, what is the likelihood of different rankings to occur? The answers follow.

4.3 Consistency probabilities : generalized results for subsets of three alternatives

This section completes Gehrlein and Fishburn (1980)'s results by computing in four-candidate elections, the limit probability (under IC) of a given ranking to occur after one candidate is removed depending on the position of the removed candidate in the original collective ranking.

With $A = \{a, b, c, d\}$, to take all the possible cases, we will consider the following collective rankings : *abcd*, *abdc*, *adbc* and *dabc*; then we suppose that d is removed and we compute the likelihood of each of the following ranking to occur : *abc*, *acb*, *bac*, *bca*, *cab*, *cba*.

The voting rules we are concerned with are those presented in Table 2.2. With $A = \{a, b, c, d\}$, we denote by $P_\infty^{IC}((w, w^B), abc/abcd)$ the limit probability under IC, that the ranking *abc* occurs when d the *last* ranked candidate of the collective ranking on $A = \{a, b, c, d\}$ is removed given the couple of tallies (w, w^B). In the same way we define $P_\infty^{IC}((w, w^B), abc/abdc)$, $P_\infty^{IC}((w, w^B), abc/adbc)$, $P_\infty^{IC}((w, w^B), abc/dabc)$, so and so. One can notice that, $P_\infty^{IC}((w, w^B), abc/abcd)$ is the probability that the three-candidate's collective ranking is consistent with the four-candidate's collective ranking when d is removed; while $P_\infty^{IC}((w, w^B), cba/abcd)$ is the probability that the three-candidate's collective ranking is totaly the reverse of that with four candidates after d is removed (as shown in Example 1).

[14] They provided the expression of the limit probability that we are not going to report here due to space constraints.

[15] This means that the collective ranking on $A = \{a, b, c, d\}$ could be *abcd* or *abdc* or *adbc* or *dabc*.

Due to space constraints, were are not going to report all the expressions of the probabilities formulas and their proofs here. But, in Section 7, we present one case in detail in Appendix A; in Appendix B we provide a MAPLE-sheet of our computation program. Tables 4.2 to 4.5 report the consistency probabilities values for each the scoring rules we analyzed in this paper.

Table 4.2: Consistency probabilities over subsets for the Borda rule on $A = \{a, b, c, d\}$

collective rankings	*abc*	*acb*	*bac*	*bca*	*cab*	*cba*
abcd	0.719518	0.138512	0.111617	0.011626	0.011888	0.006836
abdc	0.840425	0.028084	0.127008	0.002214	0.001408	0.000858
adbc	0.840425	0.127008	0.028084	0.001408	0.002214	0.000858
dabc	0.719518	0.111617	0.138512	0.011888	0.011626	0.006836
mean	0.779971	0.101305	0.101305	0.006784	0.006784	0.003847

Before going further, let us mention that, in each of the Tables 4.2 to 4.5, the first mean values computed (second column) is exactly the probability obtained by Gehrlein and Fishburn (1980); this is the mean probability that the relative ranking between three candidates is preserved when we move from a four-candidate set to its proper three-candidate subset. Our results complete those of Gehrlein and Fishburn (1980) and offer a broader outlook.

According to Tables 4.2 to 4.5, among the six scoring rules analyzed here, it comes that given a collective ranking on a four-candidate set, the Borda rule is the most consistent with the former ranking when one candidate is removed. In this, the Borda rule is followed by the Plurality extension and the Antiplurality extension, then comes the Plurality rule and the Antiplurality rule. The Limited Voting appears

Table 4.3: Consistency probabilities over subsets for the Plurality rule and the Antiplurality rule on $A = \{a, b, c, d\}$

collective rankings	*abc*	*acb*	*bac*	*bca*	*cab*	*cba*
abcd	0.516969	0.203037	0.171531	0.040395	0.042127	0.025939
abdc	0.661087	0.089474	0.213197	0.017253	0.011572	0.007415
adbc	0.661087	0.213197	0.089474	0.011572	0.017253	0.007415
dabc	0.516969	0.171531	0.203037	0.042127	0.040395	0.025939
mean	0.589028	0.169309	0.169309	0.027836	0.027836	0.016677

Table 4.4: Consistency probabilities over subsets for the Saari's Plurality extension rule and the Saari's Antiplurality extension rule on $A = \{a, b, c, d\}$

collective rankings	abc	acb	bac	bca	cab	cba
$abcd$	0.594543	0.183112	0.151478	0.026725	0.027667	0.016475
$abdc$	0.737973	0.061716	0.182233	0.008626	0.005633	0.003819
$adbc$	0.737973	0.182233	0.061716	0.005633	0.008626	0.003819
$dabc$	0.594543	0.151478	0.183112	0.027667	0.026725	0.016475
mean	0.666258	0.144635	0.144635	0.017163	0.017163	0.010147

Table 4.5: Consistency probabilities over subsets for the Limited Voting rule on $A = \{a, b, c, d\}$

collective rankings	abc	acb	bac	bca	cab	cba
$abcd$	0.412089	0.217588	0.190647	0.065521	0.068797	0.045355
$abdc$	0.537710	0.132448	0.243258	0.039356	0.027916	0.018909
$adbc$	0.537710	0.243258	0.132448	0.027916	0.039356	0.018909
$dabc$	0.412089	0.190647	0.217588	0.068797	0.065521	0.045355
mean	0.474900	0.195985	0.195985	0.050397	0.050397	0.032132

Table 4.6: Limit probability for a to remain first

collective rankings	Borda	Plurality Extension Antiplurality Extension	Plurality Antiplurality	Limited Voting
$abcd$	0.858030	0.777655	0.720006	0.629677
$abdc$	0.868509	0.799689	0.750561	0.670158
$adbc$	0.967433	0.920206	0.874284	0.780968
mean	0.897990	0.832517	0.781617	0.693601

to perform the worst. We also learn that for each of our six scoring rules, given a collective ranking on a four-candidate set, when one candidate is removed, it is more difficult under the Borda rule than under the five other voting rules to end with a total reversal ranking (or with a ranking swapping candidates) on the three-candidate set; this is more likely under the Limited Voting.

Also, we learn that, given a collective ranking, when a candidate is removed, it becomes difficult (for all of the six rules analyzed) to end with a new collective ranking in which two candidates are swapped as the gap between them increases comparatively to the original ranking. From Tables 4.2 to 4.5, we deduce for each of our scoring rules, the propensities of a top ranked candidate to remain first (see Table 4.6) or to become last (see 4.7) after a given candidate is removed.

According to Tables 4.6 and 4.7, for each of our scoring rules, when a candidate (except the top ranked one) is removed, it is more likely for a top ranked candidate to remain first (on average, 89.79% for the Borda rule, 83.25% for the extension rule, 78.16% for the Plurality rule and the Plurality and 69.36% for the Limited Voting) while it is very hard for this candidate to become last with the Borda rule (on average 0.79%; 2.16% for the extension rule, 3.67% for the Plurality rule and the Plurality and 7.01% for the Limited Voting). Also, for each of our voting rules, as candidate a and d are closed, it is very probable (resp. difficult) for a to remain first (resp. to become last) after d is removed.

We have learnt from Table 4.1 that the pairwise comparisons are more consistent with the collective ranking under the Limited Voting than under the Plurality and the Antiplurality. When considering the proper subsets of three candidates, we have the reverse. So, we can expect that, with a given a set of candidates, when studying the consistency between the collective ranking on this set and what we have on its proper subsets, the hierarchy among the scoring rules may not be the same as the size of the subsets vary.

Table 4.7: Limit probability for a to become last

collective rankings	Borda	Plurality Extension Antiplurality Extension	Plurality Antiplurality	Limited Voting
abcd	0.018462	0.043200	0.066334	0.110876
abdc	0.003072	0.012445	0.024668	0.052650
adbc	0.002266	0.009452	0.018987	0.046825
mean	0.007933	0.021699	0.036663	0.070117

According to probabilities displayed in Tables 4.2 to 4.5, we can state the following relationships between probabilities since they are valid[16] for all the scoring rules defined in Table 2.2.

Consider $A = \{a, b, c, d\}$ and each of the tallies (w, w^B) of the voting rules defined in Table 2.2,

- The limit probability that the three-candidate's collective ranking is consistent with the four-candidate's collective ranking when d is removed is as follows

$$P_{\infty}^{IC}((w, w^B), abc/abcd) = P_{\infty}^{IC}((w, w^B), abc/dabc) \tag{4.2}$$

$$P_{\infty}^{IC}((w, w^B), abc/abdc) = P_{\infty}^{IC}((w, w^B), abc/adbc) \tag{4.3}$$

- The limit probability that the three-candidate's collective ranking is totally the reverse of that with four candidates after d is removed is as follows

$$P_{\infty}^{IC}((w, w^B), cba/abcd) = P_{\infty}^{IC}((w, w^B), cba/dabc) \tag{4.4}$$

$$P_{\infty}^{IC}((w, w^B), cba/abdc) = P_{\infty}^{IC}((w, w^B), cba/adbc) \tag{4.5}$$

- Also, it comes from Tables 4.2 to 4.5 that for $(\phi, \tau) = (acb, bac)$ or (cab, bca),

$$P_{\infty}^{IC}((w, w^B), \phi/abcd) = P_{\infty}^{IC}((w, w^B), \tau/dabc) \tag{4.6}$$

$$P_{\infty}^{IC}((w, w^B), \phi/abdc) = P_{\infty}^{IC}((w, w^B), \tau/adbc) \tag{4.7}$$

$$P_{\infty}^{IC}((w, w^B), \phi/adbc) = P_{\infty}^{IC}((w, w^B), \tau/abdc) \tag{4.8}$$

$$P_{\infty}^{IC}((w, w^B), \phi/dabc) = P_{\infty}^{IC}((w, w^B), \tau/abcd) \tag{4.9}$$

[16] At this stage of the analysis, all the relationships stated remain as conjectures for all the others scoring apart of those explored in this paper. So, we need to explore all the possible vector combinations w and w^B in order to generalize these relationships. This stills a challenge.

5 Evaluating the likelihood of two paradoxes of committees elections

As already said, our probabilities complete Gehrlein and Fishburn (1980)'s results and give a complete picture on the consistency of collective rankings and those on subsets of alternatives. In this section, we will show how we can exploit the exhaustibility of our results. More particularly, we will focus on the paradoxes that can occur when one wishes to select a committee of two out of four candidates.

5.1 Electing committees

Electoral laws and constitutions of many democratic organizations and countries stipulate that, when electing a committee or a group of representatives (a parliament, a board, etc.), every elected candidate must have at least one *substitute*. A substitute is supposed to take the place of the elected candidate if for various reasons, he comes to leave. In contrast, there are some organizations that elect boards without provision of substitutes: for some of them, nothing is said on what will be done if one elected member decided to leave; for others, a recourse to new (partial) elections is required in such a case.

Suppose that a ballot with m candidates (at least three) leads to a committee made of the g candidates with the greatest scores (g at least equal to two) and that there is no provision of substitutes for any elected member of the committee. In such a case, if an elected candidate decides to leave, there are three possible ways to fill the empty chair *ceteris paribus*[17] : *Rule 1* (R_1) simply *nominate* the candidate ranked *g+1-th* in the original ballot; *Rule 2* (R_2) take a *new ballot* on the remaining $m-1$ candidates and pick the g greatest scores; *Rule 3* (R_3) take a *partial ballot* on the others $m-g$ non elected candidates and take the one who scores the best. It is clear that rule R_1 is the natural way to replace the leaving candidate. Nonetheless, the different rules can lead to different committees. How often?

In order to answer, we define and analyze the discrepancies between the Rule 1 and the two others possible ways to fill an empty chair.

5.2 Prior Successor Paradox

Since we have supposed that the elected committee is formed by the candidates with the g greatest scores (in particular with scoring rules), we define the Prior Successor

[17] This means that, voters keep their preferences unchanged on the rest of candidates no matter who leaves.

as the candidate with the *g+1-th* best scores.

Definition 1. (*Prior Successor*) Let $m \geq 3$ and $1 \leq g \leq m-1$ and consider a given scoring rule. For $C^* \in C^g$, a candidate a is called the *Prior Successor* if $a \notin C^*$ and candidate a is the one with the $g+1$-*th* greatest score.

The *Prior Successor Paradox* (PSP) occurs if after a member of the elected committee leaves, a new ballot (given the subset of candidates) leads to a committee containing all the $g-1$ members of the previous committee without the *Prior Successor.*

Definition 2. (*Prior Successor Paradox*) Let $m \geq 3$, $1 \leq g \leq m-1$ and an elected committee C^* such that candidate a is the Prior Successor ($a \notin C^*$). The *Prior Successor Paradox* occurs if when a candidate $d \in C^*$ leaves, we have $a \notin \hat{C}^*$ and $C^* \setminus \{d\} \subset \hat{C}^*$.

Thus, the PSP can occur when one chooses not to just nominate the prior successor as recommended by rule R_1. If a new ballot is taken (Rule R_2) when an elected candidate leaves, we denote by $P^{IC_\infty}_{\mathrm{PSP}_j}(w, w^B)$ the limit probability under IC of the PSP given that the leaving candidate is the one who was ranked *j-th* ($j \leq g$) in the collective ranking. This is also the limit probability that the new elected committee differs from that recommended by rule R_1. Given that the collective ranking is *dabc* and that the two-member committee is (a, d), if candidate d leaves, the $\mathrm{PSP}_1(w, w^B)$ will occur after a new ballot if the new elected two-member committee is (a, c) *i.e* the new collective ranking is *acb* or *cab*. So,

$$P^{IC_\infty}_{\mathrm{PSP}_1}(w, w^B) = P^{IC}_\infty((w, w^B), acb/dabc) + P^{IC}_\infty((w, w^B), cab/dabc) \tag{5.1}$$

If candidate a leaves, the PSP_2 will occur after a new ballot if the new elected two-member committee is (d, c) *i.e* the new collective ranking is *dcb* or *cdb*. So,

$$\begin{aligned} P^{IC_\infty}_{\mathrm{PSP}_2}(w, w^B) &= P^{IC}_\infty((w, w^B), dcb/dabc) + P^{IC}_\infty((w, w^B), cdb/dabc) \\ &= P^{IC}_\infty((w, w^B), acb/adbc) + P^{IC}_\infty((w, w^B), cab/adbc) \end{aligned} \tag{5.2}$$

For four-candidate elections and two-member committees, Table 5.1 gives the values of $P^{IC_\infty}_{\mathrm{PSP}_1}(w, w^B)$ (and that of $P^{IC_\infty}_{\mathrm{PSP}_2}(w, w^B)$) which is the probability of the PSP when the leaving candidate is the one with the highest (resp. second best) score in the elected two-member committee. According to these probabilities, among the six voting rules analyzed her, the Limited voting appears to be the most vulnerable to

Table 5.1: Likelihood of the PSP

	New ballot PSP		Partial ballot
	$P^{IC_\infty}_{\text{PSP}_1}$	$P^{IC_\infty}_{\text{PSP}_2}$	$P^{IC_\infty}_{\text{PSP}}$
Borda	0.123243	0.129222	0.200153
Plurality Antiplurality	0.211926	0.230450	0.319487
Limited Voting	0.256168	0.282614	0.268601
Plurality extended Antiplurality extended	0.178203	0.190859	0.258908

both the $P^{IC_\infty}_{\text{PSP}_1}$ and the $P^{IC_\infty}_{\text{PSP}_2}$ with respectively 25.62% and 28.26%. It is followed in this by the Plurality and the Antiplurality rule (21.19% and 23.04%) and by the extension rules proposed by Saari (17.82% and 19.08%). The Borda rule exhibits these paradoxes the less with respectively 12.32% and 12.92%. Also, for each of these six scoring rules, the $P^{IC_\infty}_{\text{PSP}_1}$ is less likely than the $P^{IC_\infty}_{\text{PSP}_2}$. Thus, in four candidate-election and for two-member committees, the PSP is less likely to occur if the candidate with the highest score in the committee leaves.

Notice that our probabilities do not allow us to derive the probability of the PSP no matter the rank of the leaving candidate. Assume for $A = \{a, b, c, d\}$ that the elected committee is $\{a, b\}$; nothing tell us that the PSP only occurs when a leaves and not when b leaves and vice-versa. So, we need more than what are displayed in our tables : the joint probabilities[18].

What do we have in case of a partial ballot (Rule R_3)? In four candidate and two-member committee, given that the collective ranking is *abcd* and that the two-member committee is (a, b), the PSP will occur with a partial ballot after a member of the elected committee leaves if and only if candidate c who is the Prior Successor is beaten in pairwise majority by candidate d. So, probability $P^{IC_\infty}_{\text{PSP}}(w, w^B)$ of the PSP in this case is the probability that the pairwise majority between the two last ranked candidates is not consistent with the collective ranking on the four candidates. It comes that, $P^{IC_\infty}_{\text{PSP}}(w, w^B)$ is the complementary of the results obtained by Gehrlein and Fishburn (1980) (see Table 4.1) on the consistency between pairwise majority and the collective ranking in four-candidate elections So, with *abcd* as the collective

[18] We admit that, with our computation techniques we are not yet able to get these joint probabilities.

ranking,

$$P_{\text{PSP}}^{IC_\infty}(w, w^B) = 1 - q((w, w^B), cd) \tag{5.3}$$

The last column of Table 5.1 gives the values of $P_{\text{PSP}}^{IC_\infty}(w, w^B)$ for each of our analyzed scoring rules. According to these values, it is the Plurality rule and the Antiplurality rule that are the most likely with 31.95% to produces the PSP in case of a partial ballot. They are followed by the Limited Voting (26.86%), the Extension rules (25.89%) and the Borda rule (20.01%).

We also deal with another paradox of committees that is more stronger than the PSP : the *Leaving Member Paradox* (LMP).

5.3 The Leaving Member Paradox

According to Staring (1986), when voters vote for exactly g candidates in order to elect a board of size[19] g , if a new ballot is taken, *ceteris paribus*, after an elected candidate leaves (this is consistent with (R_2)), we can end with a committee that does not contain one or many members of the original one even worse, the two committees may be disjoint : this is the *Leaving Member Paradox*(LMP)[20]. The formal definition follows.

Suppose that we want to elect a committee of g members ($2 \leq g < m - 1$) and that the elected candidates are those with the g greatest scores. We denote by C^g the set of all possible committees of size g, by $C^* \in C^g$ the elected committee. If a candidate leaves C^*, we denote by $\hat{C}^* \in C^g \setminus C^*$ the new elected committee after a ballot has be taken to replace this leaving candidate.

Definition 3. (*Leaving Member Paradox*) For $m \geq 4$ and $2 \leq g \leq m - 1$, consider a given scoring rule. The *Leaving Member Paradox* occurs if for $C^*, \hat{C}^* \in C^g$, $\exists a \in C^*$ and a leaving member $d \in C^* \setminus \{a\}$ such that $a \notin \hat{C}^*$.

Since we have taken $2 \leq g \leq m - 1$, it is clear that with three candidates, we can only elect a committee of size two; so, if a member leaves, there is no reason for the Leaving Member Paradox to happen. Notice that the LMP is not defined with less than four candidates.

[19]This defines the Limited Voting.

[20]The other paradox described by Staring (1986) is the *Increasing Committee Size Paradox* which occurs if an elected member of a committee of size g is no more elected for a committee of size $g+1$; even worse, both committees may be totally disjoint. For more about the *Increasing Committee Size Paradox*, see Staring (1986), Mitchell and Trumbull (1992). We only concern here with the LMP which is defined as it follows.

Recall that the LMP follows the recommendation of rule R_2; so by following the recommendation of rule R_2, we can end with a committee totaly different from what we will have by rule R_1. If a new ballot is taken when an elected candidate leaves, we denote by $P_{\text{LMP}_j}^{IC_\infty}(w, w^B)$ the limit probability under IC of the LMP given that the leaving candidate is the one who was ranked *j-th* ($j \leq g$) in the collective ranking. Given that the collective ranking is *dabc* and that the two-member committee is (a, d), if candidate d leaves, the LMP_1 will occur if the new elected two-member committee is (b, c) *i.e* the new collective ranking is *bca* or *cba*. So,

$$P_{\text{LMP}_1}^{IC_\infty}(w, w^B) = P_\infty^{IC}((w, w^B), bca/dabc) + P_\infty IC((w, w^B), cba/dabc) \tag{5.4}$$

If candidate a leaves, the LMP_2 will occur if the new elected two-member committee is (d, c) *i.e* the new collective ranking is *bcd* or *cbd*. So,

$$\begin{aligned} P_{\text{LMP}_2}^{IC_\infty}(w, w^B) &= P_\infty^{IC}((w, w^B), bcd/dabc) + P_\infty^{IC}((w, w^B), cbd/dabc) \\ &= P_\infty^{IC}((w, w^B), bca/adbc) + P_\infty^{IC}((w, w^B), cba/adbc) \end{aligned} \tag{5.5}$$

Table 5.2: Likelihood of the LMP_1 and the LMP_2

	New ballot	
	LMP	
	$P_{\text{LMP}_1}^{IC_\infty}$	$P_{\text{LMP}_2}^{IC_\infty}$
Borda	0.018724	0.002266
Plurality Antiplurality	0.068076	0.018987
Limited Voting	0.114152	0.046825
Plurality extended Antiplurality extended	0.044142	0.009452

For four-candidate elections and two-member committees, Table 5.2 gives the values of $P_{\text{LMP}_1}^{IC_\infty}(w, w^B)$ (and that of $P_{\text{LMP}_2}^{IC_\infty}(w, w^B)$) which is the probability of the LMP when the leaving candidate is the one with the highest (resp. second best) score in the elected two-member committee. According to these probabilities, among the six voting rules analyzed here, the Limited voting appears to be the most vulnerable to both the $P_{\text{LMP}_1}^{IC}$ and the $P_{\text{LMP}_2}^{IC}$ with respectively 11.41% and 4.68%. It is followed in this by the Plurality and the Antiplurality rule (6.81% and 1.89%) and by the extension rules proposed by Saari (4.41% and 0.95%). The Borda rule exhibits these paradoxes the less with respectively 1.87% and 0.23%. Also, for each of these six scoring rules, the $P^{IC_\infty}_{\mathrm{LMP}_1}$ is more likely than the $P^{IC}_{\mathrm{LMP}_2}$. Thus, in four candidate-election and for two-member committees, the LMP is more likely to occur if the candidate with the highest score in the committee leaves.

For the same reasons we gave for the PSP, our probabilities do not allow us to derive the probability of the LMP no matter the ranking of the leaving candidate.

6 Conclusion

The main objective of this paper was to revisit a classical theme in Social Choice Theory, the stability of the collective ranking as candidates leave or enter the choice set. We focussed on the analysis of the discrepancies between the rankings on four-candidate sets and three-candidate sets obtained by scoring rules, by deriving the exact values of the probabilities of all the possible scenarios under the Impartial Culture assumption. We provided results not only for classical rules, such as the Borda count, the simple Plurality rule, and the Antiplurality rule, but also for less known rules, such as the Limited Voting, the Plurality extension and the Antiplurality extension. Our results complete previous analysis by Gehrlein and Fishburn (1980) on the consistency of the rankings obtained by scoring rules on subsets of candidates. As a by product, we could easily estimate from our tables the likelihood of some paradoxical events for the election of a two-member committee from a four candidate menu. We showed that the different ways to fill an empty set after an elected member leaves (by picking the prior successor, organizing a partial ballot or starting the whole electoral process again) could lead to very different choices quite frequently for some rules.

Our study confirms the superiority of the Borda count within the class of scoring rules under the Impartial Culture assumption when it comes to minimize the likelihood of discrepancies across subsets of candidates. This fact was already well documented for pairwise majorities; we now extend this result for relationships between four candidates and three candidate rankings. Moving up to four candidates also permits us to study rules for which only few things are known, like Limited Voting. It turns out that it performs quite badly. In particular, it is unable to keep a ranking in a three candidate subset in more than 50% of the cases (see Table 4.5) while the other five scoring rules we study do better, up to 78% for the Borda count. However, we also observe that obtaining a clear cut ranking of the scoring rules on their propensity to respect orderings is not possible: the Limited Voting does better than Plurality and Antiplurality when it comes to satisfy pairwise ranking.

The four-candidate case also enabled us to study Saari's Plurality extension and Antiplurality extension. Though they cannot beat the Borda Count, they clearly perform better than Plurality, Antiplurality and Limited Voting, not only when comparing the three candidate ranking with the four-candidate ranking, but also on the basis on pairwise comparisons. Hence using the pair $(w = (1, \frac{1}{3}, 0, 0), w^B = (1, 0, 0)$ could be viewed as an alternative to classical simple Plurality when one wishes to clearly give more points to its first choice.

When it comes to the selection of a committee, our study compared three ways to fill the empty seat : selecting the prior successor, running a partial election, or starting the electoral process all over again. Though filling the empty seat with the successor is the easiest and costliest method, the figures we observe for the Prior Successor Paradox tell us it may lead us to a wrong choice quite frequently. This paradox is especially large if we use the Plurality rule or the Antiplurality rule: the prior successor could be defeated in almost 32% of the case in a partial ballot, and in more than 20% of the case with a full slate election. This figures suggests that, in many cases, it would be a better options to vote again in order to stick to the wishes of the voters. The values we derived for the Leaving Member Paradox teach us that, electing a completely new committee after the departure of a seating member remains a rare event.

At this stage, one may wonder why we did not estimate the discrepancies between the partial ballot method and the full ballot method. To obtain these figures, one needs to estimate the likelihoods of words on, let's say, $\{a, b, c, d\}$, $\{a, b, c\}$ and $\{b, c\}$. To describe a precise word on these subsets, one needs at least six inequalities. Unfortunately, there does not exist general techniques that enable us to derive the exact formulas of the likelihood of such an event under IC. For this reason too, we had to distinguish between LMP_1 and LMP_2, as the we cannot evaluate the likelihood of the joint event. For the same reason, we had to distinguish between PSP_1 and PSP_2 with the new ballot approach. At the moment, one should rely on Monte-Carlo simulations to obtain an estimation of the likelihood of these events.

Hence, though we had a glance at the relationships between the rankings on size three and size four subsets, and at the probabilities of some related events, there still many question to analyze when we consider more alternatives. As said before, deriving precise formulas for the likelihood of events described by more than five inequalities is almost impossible. Though one may conjecture that the Borda count will minimize the likelihood of inconsistencies on subsets for any number of candidates, a precise proof of this statement is beyond reach. Having more alternatives opens the door to the use of new voting rules. We already discussed Limited Voting, the Plurality extension and the Antiplurality extension. But, we now almost nothing about the likelihood of paradoxical events for the rules of k-names, which are frequently used for the election of the committees of size k. More generally, the rule of k-names simply ask each voter to write k names on his ballot, and to give one point to each candidate marked on a ballot. The poor performance of Limited Voting in our setting cast a doubt on the use of such rules. To conclude, only massive simulations and/or the development of new computation techniques will enable us to precisely estimate the likelihood of many paradoxes when more than four candidates are in play.

7 Appendices : Details on probability computations

7.1 Appendix A : One case in detail

In this section we will only focus on the probability computation of the event "*abcd* is the collective ranking of the Plurality rule and *abc* is the new collective ranking when d is removed". The calculations for all the other events and for all the others scoring rules follow the same scheme. So, with $A = \{a, b, c, d\}$ and $B = \{a, b, c\}$, our objective is to evaluate the probability of the event described by equation (7.1) under the IC assumption, for n large.

$$\begin{cases} S(A, w_P, \pi, a) > S(A, w_P, \pi, b) \\ S(A, w_P, \pi, b) > S(A, w_P, \pi, c) \\ S(A, w_P, \pi, c) > S(A, w_P, \pi, d) \\ S(B, w_P, \pi, a) > S(B, w_P, \pi, b) \\ S(B, w_P, \pi, b) > S(B, w_P, \pi, c) \end{cases} \tag{7.1}$$

Given the labels of the 24 preference types for $m = 4$ presented in Table 2.1, we rewrite each of the equations. $S(A, w_P, \pi, a) > S(A, w_P, \pi, b)$ is described by equation (7.2):

$$n_1 + n_2 + n_3 + n_4 + n_5 + n_6 > n_7 + n_8 + n_9 + n_{10} + n_{11} + n_{12} \tag{7.2}$$

$S(A, w_P, \pi, b) > S(A, w_P, \pi, c)$ is described by equation (7.3):

$$n_7 + n_8 + n_9 + n_{10} + n_{11} + n_{12} > n_{13} + n_{14} + n_{15} + n_{16} + n_{17} + n_{18} \tag{7.3}$$

$S(A, w_P, \pi, c) > S(A, w_P, \pi, d)$ is described by equation (7.4):

$$n_{13} + n_{14} + n_{15} + n_{16} + n_{17} + n_{18} > n_{19} + n_{20} + n_{21} + n_{22} + n_{23} + n_{24} \tag{7.4}$$

$S(B, w_P, \pi, a) > S(B, w_P, \pi, b)$ is described by equation (7.5):

$$n_1+n_2+n_3+n_4+n_5+n_6+n_{19}+n_{20} > n_7+n_8+n_9+n_{10}+n_{11}+n_{12}+n_{21}+n_{22} \quad (7.5)$$

$S(B, w_P, \pi, b) > S(B, w_P, \pi, c)$ is described by equation (7.6):

$$n_7+n_8+n_9+n_{10}+n_{11}+n_{12}+n_{21}+n_{22} > n_{13}+n_{14}+n_{15}+n_{16}+n_{17}+n_{18}+n_{23}+n_{24} \quad (7.6)$$

To make our calculations, we proceed by two steps.

1st step: we compute the probability of the event "$abcd$ is the collective ranking and $S(B, w_P, \pi, a) > S(B, w_P, \pi, b)$ when d is removed"

This event is described by equations (7.2 to 7.5). To do so, we rewrite equation 7.5 by using a parameter t.

$$n_1+n_2+n_3+n_4+n_5+n_6+tn_{19}+tn_{20} > n_7+n_8+n_9+n_{10}+n_{11}+n_{12}+tn_{21}+tn_{22} \quad (7.7)$$

When $t = 0$, equation (7.7) is equivalent to equation (7.2). In $t = 1$, it fully describes the situation $S(B, w_P, \pi, a) > S(B, w_P, \pi, b)$. Our proof technique will in fact evaluate the probability that equations (7.2),(7.3),(7.4) and (7.7) are satisfied under IC for n large. In $t = 0$, we recover the value $\frac{1}{24}$ which is the probability of $abcd$ to be the collective ranking while in $t = 1$, we will derive the probability the event described by equations (7.2),(7.3),(7.4) and (7.7).

Given four candidates, it is assumed under the Impartial Culture assumption that each voter is equally likely to have one of the 24 preference types. Let x_i be the random variable that associates to each voter i a 24-component vector with probability $\frac{1}{24}$ of having 1 in each position. The expectation of x_i is

$$E(x_i) = \left(\frac{1}{24}, \frac{1}{24}, \ldots, \frac{1}{24}\right)$$

and the covariance matrix is a diagonal 24×24 matrix with the common entry σ given by

$$\sigma^2 = E(x_i^2) - E(x_i)^2$$

Let

$$m^T = (m_1, m_2, \ldots m_{24})^T = \frac{1}{\sigma\sqrt{n}}\left[\begin{pmatrix} n_1 \\ \vdots \\ n_{24} \end{pmatrix} - \begin{pmatrix} \frac{n}{24} \\ \vdots \\ \frac{n}{24} \end{pmatrix}\right]$$

The Central Limit Theorem in $\mathbb{R}^{23}$ implies

$$\mu\left[m^T\right] \mapsto \frac{1}{(\sqrt{2}\pi)^{23}} e^{\frac{-|t|^2}{2}} \lambda$$

as $n \to \infty$ where $t = (t_1, t_2, \ldots, t_{24}) \in \mathbb{R}^{24}$, $|t|^2 = t_1^2 + \cdots + t_{24}^2$ and λ is the Lebesgue measure on the 23-dimensional hyperplane $t_1 + \cdots + t_{24} = 0$. Note that since m^T has the measure supported on the hyperplane $m_1 + \cdots + m_{24} = 0$, the limit of m^T as $n \to \infty$ is also a measure supported on $t_1 + \cdots + t_{24} = 0$. To compute the probability that $abcd$ is the collective ranking and $S(B, w_P, \pi, a) > S(B, w_P, \pi, b)$ when d is removed, we need to evaluate the probability that a voting situation is characterized by the inequalities (7.2),(7.3),(7.4) and (7.7); m satisfies inequalities (7.2),(7.3),(7.4) and (7.7) if and only if $\tilde{n} = (n_1, n_2, \ldots, n_{24})$ also satisfies them. Then, by the Central Limit Theorem, we write

$$Pr\left(m^T \quad \text{satisfies} \quad (7.2), (7.3), (7.4) \quad \text{and} \quad (7.7)\right) \mapsto \frac{1}{(\sqrt{2}\pi)^{23}} \int_{C_1} e^{\frac{-|t|^2}{2}} d\lambda$$

where $C_1 = \{t \in \mathbb{R}^{24}, t \text{ satisfies } ((7.2), (7.3), (7.4) \quad \text{and} \quad (7.7); \text{and } \sum_{i=1}^{24}(t_i) = 0\}$.

As the measure

$$\bar{\mu} \equiv \frac{1}{(\sqrt{2}\pi)^{23}} e^{\frac{-|t|^2}{2}} \lambda$$

is absolutely continuous and radially symmetric, computing

$$\frac{1}{(\sqrt{2}\pi)^{23}} \int_{C} e^{\frac{-|t|^2}{2}} d\lambda$$

reduces to finding the measure $\bar{\mu}$ of the cone C_1, when the measure is invariant to rotations. The measure $\bar{\mu}$ of such a cone is proportional to the Euclidean measure of the cone, that is, the measure on the sphere.

Saari and Tataru (1999) have developed a method of computing the probabilities of voting events under the Impartial culture. Some refinements of this method are done in Merlin *et al.* (2000), Merlin and Valognes (2004). This method is mainly based on linear algebra and the calculation of a differential volume in a spherical simplex of dimension ν using the Schläfli's formula (See Coxeter (1935), Schläfli (1950), Milnor (1982), Kellerhals (1989)). This formula is given by:

$$dvol_\nu(C_1) = \frac{1}{(\nu - 1)} \sum_{0 \leq j < k \leq \nu} vol_{\nu-2}(S_j \cap S_k) d\alpha_{jk}; \quad vol_0 = 1$$

with α_{jk} the dihedral angle formed by the facets S_j and S_k of the cone C_1. Following the arguments given by Saari and Tataru (1999), the probability that these inequalities are met simultaneously for a voting situation when $p_i = \frac{1}{24},\ i = 1, \ldots, 24$ for n large is equal to the surface of the spherical simplex T described by equations (7.2),(7.3),(7.4), (7.7) on the surface of the unit sphere in $\mathbb{R}^4$, divided by the surface of this sphere. More precisely, if we denote by $P_\infty^{IC}((w, w^B), abcd - ab)$ the limit probability that $abcd$ is the collective ranking and $S(B, w_P, \pi, a) > S(B, w_P, \pi, b)$ when d is removed, we will derive

$$P_\infty^{IC}((w, w^B), abcd - ab) = 1 + \frac{1}{\omega^4} \int_0^t dvol_\nu(C_1)$$

where $\omega^4 = 2\pi^2$ is the volume of the unit sphere in $\mathbb{R}^4$.

Given the cone C_1, let S_1 be the facet defined by the equation (7.2), S_2 the facet defined by the equation (7.3), S_3 the facet defined by the equation (7.4) and S_4 the facet defined by the equation (7.7).

Let $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4$ be the normal vectors to the hyperplanes S_1, S_2, S_3, S_4.

$$\begin{aligned}
\vec{v_1} &= (1,1,1,1,1,1,-1,-1,-1,-1,-1,-1,0,0,0,0,0,0,0,0,0,0,0,0) \\
\vec{v_2} &= (0,0,0,0,0,0,1,1,1,1,1,1,-1,-1,-1,-1,-1,-1,0,0,0,0,0,0) \\
\vec{v_3} &= (0,0,0,0,0,0,0,0,0,0,0,0,1,1,1,1,1,1,-1,-1,-1,-1,-1,-1) \\
\vec{v_4} &= (1,1,1,1,1,1,-1,-1,-1,-1,-1,-1,0,0,0,0,0,0,t,t,-t,-t,0,0)
\end{aligned}$$

Since $\vec{v_j}$ and $\vec{v_k}$ are respectively normal to S_j and S_k, we can use the relationship

$$cos(\alpha_{jk}) = \frac{-\vec{v_j}.\vec{v_k}}{||\vec{v_j}||.||\vec{v_k}||}$$

to derive the value of the dihedral angle α_{jk} between vectors $\vec{v_j}$ and $\vec{v_k}$.

$$\begin{aligned}
&\alpha_{12} = \alpha_{23} = \tfrac{\pi}{3} \\
&\alpha_{13} = \alpha_{34} = \tfrac{\pi}{2} \\
&\alpha_{14} = \pi - \arccos\left(\tfrac{\sqrt{3}}{\sqrt{3+t^2}}\right) \\
&\alpha_{24} = \arccos\left(\tfrac{\sqrt{3}}{2\sqrt{3+t^2}}\right)
\end{aligned}$$

Therefore,

$$\begin{aligned}
&d\alpha_{12} = d\alpha_{13} = d\alpha_{23} = d\alpha_{34} = 0 \\
&d\alpha_{14} = \tfrac{-\sqrt{3}}{3+t^2} \\
&d\alpha_{24} = \tfrac{t\sqrt{3}}{(3+t^2)\sqrt{9+4t^2}}
\end{aligned}$$

The vectors $\vec{v_1}, \vec{v_2}, \vec{v_3}, \vec{v_4}$ lie in a 4-dimension space. Vectors $\vec{v_5}$ to $\vec{v_{24}}$ form a basis for the orthogonal subspace:

$$
\begin{aligned}
\vec{v}_5 &= (1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1)\\
\vec{v}_6 &= (-1,0,0,0,0,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0)\\
\vec{v}_7 &= (-1,0,0,0,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0)\\
\vec{v}_8 &= (-1,0,0,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0)\\
\vec{v}_9 &= (-1,0,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0)\\
\vec{v}_{10} &= (-1,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0)\\
\vec{v}_{11} &= (0,0,0,0,0,0,-1,0,0,0,0,1,0,0,0,0,0,0,0,0,0,0,0,0)\\
\vec{v}_{12} &= (0,0,0,0,0,0,-1,0,0,0,1,0,0,0,0,0,0,0,0,0,0,0,0,0)\\
\vec{v}_{13} &= (0,0,0,0,0,0,-1,0,0,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0)\\
\vec{v}_{14} &= (0,0,0,0,0,0,-1,0,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0)\\
\vec{v}_{15} &= (0,0,0,0,0,0,-1,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0)\\
\vec{v}_{16} &= (0,0,0,0,0,0,0,0,0,0,0,0,-1,0,0,0,0,1,0,0,0,0,0,0)\\
\vec{v}_{17} &= (0,0,0,0,0,0,0,0,0,0,0,0,-1,0,0,0,1,0,0,0,0,0,0,0)\\
\vec{v}_{18} &= (0,0,0,0,0,0,0,0,0,0,0,0,-1,0,0,1,0,0,0,0,0,0,0,0)\\
\vec{v}_{19} &= (0,0,0,0,0,0,0,0,0,0,0,0,-1,0,1,0,0,0,0,0,0,0,0,0)\\
\vec{v}_{20} &= (0,0,0,0,0,0,0,0,0,0,0,0,-1,1,0,0,0,0,0,0,0,0,0,0)\\
\vec{v}_{21} &= (0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,-1,1,0,0,0,0)\\
\vec{v}_{22} &= (0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,-1,1)\\
\vec{v}_{23} &= (0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1,0,0,1,-2,0)\\
\vec{v}_{24} &= (0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1,0,1,0,-2,0)
\end{aligned}
$$

Then, we can calculate the vertexes $P_{123} = S_1 \cap S_2 \cap S_3$, $P_{124} = S_1 \cap S_2 \cap S_4$, $P_{134} = S_1 \cap S_3 \cap S_4$ and $P_{2345} = S_2 \cap S_3 \cap S_4$ by solving the following systems

$$
P_{123} : \begin{cases} S_1 = 0 \\ S_2 = 0 \\ S_3 = 0 \\ S_4 > 0 \\ S_5 = 0 \\ S_6 = 0 \\ S_7 = 0 \\ \vdots \\ \vdots \\ S_{23} = 0 \\ S_{24} = 0 \end{cases}
\quad
P_{124} : \begin{cases} S_1 = 0 \\ S_2 = 0 \\ S_3 > 0 \\ S_4 = 0 \\ S_5 = 0 \\ S_6 = 0 \\ S_7 = 0 \\ \vdots \\ \vdots \\ S_{23} = 0 \\ S_{24} = 0 \end{cases}
\quad
P_{134} : \begin{cases} S_1 = 0 \\ S_2 > 0 \\ S_3 = 0 \\ S_4 = 0 \\ S_5 = 0 \\ S_6 = 0 \\ S_7 = 0 \\ \vdots \\ \vdots \\ S_{23} = 0 \\ S_{24} = 0 \end{cases}
\quad
P_{234} : \begin{cases} S_1 > 0 \\ S_2 > 0 \\ S_3 = 0 \\ S_4 = 0 \\ S_5 = 0 \\ S_6 = 0 \\ S_7 = 0 \\ \vdots \\ \vdots \\ S_{23} = 0 \\ S_{24} = 0 \end{cases}
$$

The solutions of theses systems are:

$$
\begin{array}{rcl}
P_{123} & = & (0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1,1,-1,-1,0,0) \\
P_{124} & = & (1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,-3,-3,-3,-3,-3,-3) \\
P_{134} & = & (1,1,1,1,1,1,1,1,1,1,1,1,-1,-1,-1,-1,-1,-1,-1,-1,-1,-1,-1,-1) \\
P_{234} & = & (3t,3t,3t,3t,3t,3t,-t,-t,-t,-t,-t,-t,-t,-t,-t,-t,-t,-t,-6-t, \\
 & & \quad -6-t,6-t,6-t,-t,-t)
\end{array}
$$

Knowing these vertices, we are able to compute the volumes

$$
\begin{array}{l}
vol(S_1 \cap S_2) = vol(S_1 \cap S_3) = \frac{\pi}{2} \\
vol(S_1 \cap S_4) = arccos(\frac{\sqrt{3}}{3}) \\
vol(S_2 \cap S_3) = \pi - arccos(\frac{2}{\sqrt{2t^2+4}}) \\
vol(S_2 \cap S_4) = arccos(\frac{t}{3\sqrt{t^2+2}}) \\
vol(S_3 \cap S_4) = arccos(\frac{t}{\sqrt{3t^2+6}})
\end{array}
$$

It comes from the Schläfli's formula that,

$$
\begin{array}{rcl}
dvol(C_1) & = & 2vol(S_1 \cap S_2)d\alpha_{13} + vol(S_1 \cap S_4)d\alpha_{14} + vol(S_2 \cap S_3)d\alpha_{23} \\
 & & +vol(S_2 \cap S_4)d\alpha_{24} + vol(S_3 \cap S_4)d\alpha_{34}
\end{array}
$$

We have to multiply $dvol(C_1)$ by 24 and divide it 2 (since $\nu = 3$) and by $2\pi^2$ the volume of the unit sphere in $\mathbb{R}^4$, then we obtain the final differential volume $\frac{24}{4\pi^2}\int_0^t dvol(C_1)dt$. We then derive at $t = 1$, the value of the probability that $abcd$ is the collective ranking and $S(B, w_P, \pi, a) > S(B, w_P, \pi, b)$ when d is removed

$$
P_\infty^{IC}((w, w^B), abcd - ab) = 1 + \frac{6}{\pi^2}\int_0^1 dvol(C_1)dt = 0.762134
$$

We can now move to the second step where we will derive our consistency probability between $abcd$ and abc when d is removed.

2nd step: we derive the consistency probability

The consistency probability we are looking for is equal to sum of $P^{IC}((w, w^B), abcd - ab)$ and the value of the probability that a voting situation is characterized by the inequalities (7.2),(7.3),(7.4), (7.5) and (7.6). To compute the probability of this voting situation, we need to rewrite inequality (7.6) using the parameter t.

$$
n_7+n_8+n_9+n_{10}+n_{11}+n_{12}+tn_{21}+tn_{22} > n_{13}+n_{14}+n_{15}+n_{16}+n_{17}+n_{18}+tn_{23}+tn_{24} \tag{7.8}
$$

When $t=0$, equation (7.8) is equivalent to equation (7.3). In $t=1$, it fully describes the situation of equation (7.1). As in the first step, our proof technique will be based on the same arguments and will evaluate the probability that equations (7.2),(7.3),(7.4),(7.5) and (7.8) are satisfied under IC for n large. In $t=0$, we recover the value $P_\infty^{IC}((w,w^B),abcd-ab)$ while in $t=1$, we will derive the probability of the event described by equations (7.2),(7.3),(7.4),(7.5) and (7.8).

By the Central Limit Theorem, we can write

$$Pr\left(m^T \quad \text{satisfies} \quad (7.2),(7.3),(7.4),(7.5) \quad \text{and} \quad (7.8)\right) \mapsto \frac{1}{(\sqrt{2}\pi)^{23}}\int_{C_2} e^{\frac{-|t|^2}{2}}d\lambda$$

where $C_2=\{t\in\mathbb{R}^{24}, t$ satisfies $((7.2),(7.3),(7.4),(7.5)$ and $(7.8);$ and $\sum_{i=1}^{24}(t_i)=0\}$. Using the Schläfli's formula

$$dvol_\rho(C_2)=\frac{1}{(\rho-1)}\sum_{0\leq j<k\leq\rho} vol_{\rho-2}(T_j\cap T_k)d\theta_{jk};\quad vol_0=1$$

Following the arguments stated by Saari and Tataru (1999), the probability that inequalities (7.2),(7.3),(7.4), (7.5) and (7.8) are met simultaneously for a voting situation when $p_i=\frac{1}{24}$, $i=1,\ldots,24$ for n large is equal to the surface of the spherical simplex T described by equations (7.2),(7.3),(7.4), (7.7) on the surface of the unit sphere in $\mathbb{R}^5$, divided by the surface of this sphere. More precisely, we will derive

$$P_\infty^{IC}((w_P,w_P^B),abc/abcd)=P_\infty^{IC}((w,w^B),abcd-ab)+\frac{1}{\omega^5}\int_0^t dvol_\rho(C_2)$$

where $\omega^5=\frac{8\pi^2}{3}$ is the volume of the surface of the unit sphere in $\mathbb{R}^5$.

Given the cone C_2, let T_1 be the facet defined by the equation (7.2), T_2 the facet defined by the equation (7.3), T_3 the facet defined by the equation (7.4), T_4 the facet defined by the equation (7.5) and T_5 the facet defined by the equation (7.8).

Let $\vec{s}_1,\vec{s}_2,\vec{s}_3,\vec{s}_4,\vec{s}_5$ be the normal vectors to the hyperplanes T_1,T_2,T_3,T_4,T_5.

$$\begin{aligned}
\vec{s}_1=\vec{v_1} &= (1,1,1,1,1,1,-1,-1,-1,-1,-1,-1,0,0,0,0,0,0,0,0,0,0,0,0)\\
\vec{s}_2=\vec{v_2} &= (0,0,0,0,0,0,1,1,1,1,1,1,-1,-1,-1,-1,-1,-1,0,0,0,0,0,0)\\
\vec{s}_3=\vec{v_3} &= (0,0,0,0,0,0,0,0,0,0,0,0,1,1,1,1,1,1,-1,-1,-1,-1,-1,-1)\\
\vec{s}_4 &= (1,1,1,1,1,1,-1,-1,-1,-1,-1,-1,0,0,0,0,0,0,1,1,-1,-1,0,0)\\
\vec{s}_5 &= (0,0,0,0,0,0,1,1,1,1,1,1,-1,-1,-1,-1,-1,-1,0,0,t,t,-t,-t)
\end{aligned}$$

Added to those obtained before, we then have the dihedral angles θ_{jk} between vectors $\vec{s_j}$ and $\vec{s_k}$:

$$
\begin{aligned}
&\theta_{14} = \tfrac{5\pi}{6} \\
&\theta_{34} = \tfrac{\pi}{2} \\
&\theta_{15} = \theta_{35} = \arccos\left(\tfrac{\sqrt{3}}{2\sqrt{3+t^2}}\right) \\
&\theta_{24} = \arccos\left(\tfrac{\sqrt{3}}{4}\right) \\
&\theta_{25} = \pi - \arccos\left(\tfrac{\sqrt{3}}{\sqrt{3+t^2}}\right) \\
&\theta_{45} = \arccos\left(\tfrac{3+t}{4\sqrt{3+t^2}}\right)
\end{aligned}
$$

And then,

$$
\begin{aligned}
&d\theta_{14} = d\theta_{24} = d\theta_{34} = 0 \\
&d\theta_{15} = d\theta_{35} = d\alpha_{24} \\
&d\theta_{25} = d\alpha_{14} \\
&d\theta_{45} = \tfrac{(t-1)\sqrt{3}}{(3+t^2)\sqrt{5t^2-2t+13}}
\end{aligned}
$$

The vectors $\vec{s_1}, \vec{s_2}, \vec{s_3}, \vec{s_4}, \vec{s_5}$ lie in a 4-dimension space. Vectors $\vec{s_6}$ and to $\vec{s_{24}}$ form a basis for the orthogonal subspace:

$$
\begin{aligned}
\vec{s_6} &= (1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1) \\
\vec{s_7} &= (-1,0,0,0,0,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0) \\
\vec{s_8} &= (-1,0,0,0,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0) \\
\vec{s_9} &= (-1,0,0,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0) \\
\vec{s_{10}} &= (-1,0,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0) \\
\vec{s_{11}} &= (-1,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0) \\
\vec{s_{12}} &= (0,0,0,0,0,0,-1,0,0,0,0,1,0,0,0,0,0,0,0,0,0,0,0,0) \\
\vec{s_{13}} &= (0,0,0,0,0,0,-1,0,0,0,1,0,0,0,0,0,0,0,0,0,0,0,0,0) \\
\vec{s_{14}} &= (0,0,0,0,0,0,-1,0,0,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0) \\
\vec{s_{15}} &= (0,0,0,0,0,0,-1,0,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0) \\
\vec{s_{16}} &= (0,0,0,0,0,0,-1,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0) \\
\vec{s_{17}} &= (0,0,0,0,0,0,0,0,0,0,0,0,-1,0,0,0,0,1,0,0,0,0,0,0) \\
\vec{s_{18}} &= (0,0,0,0,0,0,0,0,0,0,0,0,-1,0,0,0,1,0,0,0,0,0,0,0) \\
\vec{s_{19}} &= (0,0,0,0,0,0,0,0,0,0,0,0,-1,0,0,1,0,0,0,0,0,0,0,0) \\
\vec{s_{20}} &= (0,0,0,0,0,0,0,0,0,0,0,0,-1,0,1,0,0,0,0,0,0,0,0,0) \\
\vec{s_{21}} &= (0,0,0,0,0,0,0,0,0,0,0,0,-1,1,0,0,0,0,0,0,0,0,0,0) \\
\vec{s_{22}} &= (0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,-1,1,0,0,0,0) \\
\vec{s_{23}} &= (0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,-1,1,0,0) \\
\vec{s_{24}} &= (0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,-1,1)
\end{aligned}
$$

Then, we can calculate the vertexes $P_{1234} = T_1 \cap T_2 \cap T_3 \cap T_4$, $P_{1235} = T_1 \cap T_2 \cap T_3 \cap T_5$,

$P_{1245} = T_1 \cap T_2 \cap T_4 \cap T_5$, $P_{1345} = T_1 \cap T_3 \cap T_4 \cap T_5$ and $P_{2345} = T_2 \cap T_3 \cap T_4 \cap T_5$ by solving the following systems

$$P_{1234} : \begin{cases} T_1 = 0 \\ T_2 = 0 \\ T_3 = 0 \\ T_4 = 0 \\ T_5 > 0 \\ T_6 = 0 \\ T_7 = 0 \\ \vdots \\ \vdots \\ T_{23} = 0 \\ T_{24} = 0 \end{cases} \quad P_{1235} : \begin{cases} T_1 = 0 \\ T_2 = 0 \\ T_3 = 0 \\ T_4 > 0 \\ T_5 = 0 \\ T_6 = 0 \\ T_7 = 0 \\ \vdots \\ \vdots \\ T_{23} = 0 \\ T_{24} = 0 \end{cases} \quad P_{1245} : \begin{cases} T_1 = 0 \\ T_2 = 0 \\ T_3 > 0 \\ T_4 = 0 \\ T_5 = 0 \\ T_6 = 0 \\ T_7 = 0 \\ \vdots \\ \vdots \\ T_{23} = 0 \\ T_{24} = 0 \end{cases} \quad P_{1345} : \begin{cases} T_1 = 0 \\ T_2 > 0 \\ T_3 = 0 \\ T_4 = 0 \\ T_5 = 0 \\ T_6 = 0 \\ T_7 = 0 \\ \vdots \\ \vdots \\ T_{23} = 0 \\ T_{24} = 0 \end{cases} \quad P_{2345} : \begin{cases} T_1 > 0 \\ T_2 = 0 \\ T_3 = 0 \\ T_4 = 0 \\ T_5 = 0 \\ T_6 = 0 \\ T_7 = 0 \\ \vdots \\ \vdots \\ T_{23} = 0 \\ T_{24} = 0 \end{cases}$$

The solutions of theses systems are:

$$\begin{aligned} P_{1234} &= (0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1,1,1,1,-2,-2) \\ P_{1235} &= (0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,2,2,-1,-1,-1,-1) \\ P_{1245} &= (1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,-3,-3,-3,-3,-3,-3) \\ P_{1345} &= (t,t,t,t,t,t,t,t,t,t,t,t,-t,-t,-t,-t,-t,-t,-2-t,-2-t,-2-t, \\ &\qquad -2-t,4-t,4-t) \\ P_{2345} &= (3,3,3,3,3,3,-1,-1,-1,-1,-1,-1,-1,-1,-1,-1,-1,-1,-9,-9,3,3,3,3) \end{aligned}$$

Knowing these vertices, we are able to compute the volumes $(T_j \cap T_k)$. Each of these volumes is the area of a triangle on the sphere in $\mathbb{R}^3$ defined by some directions. Table 7.1 gives the direction for each of these volumes. Let us consider the volume $(T_1 \cap T_5)$. By the Gauss-Bonnet theorem, the area of the triangle on the sphere in R^3 defined by directions P_{1235}, P_{1245} and P_{1345} is equal to the sum of the angles on the surface of the triangle minus π. We denote by γ_{1235}, γ_{1245} and γ_{1345} the angles on the surface of the triangle respectively defined at the vertexes P_{1235}, P_{1245} and P_{1345}. Also, we define the angles $\delta_1 = \widehat{P_{1235}, P_{1245}}$, $\delta_2 = \widehat{P_{1235}, P_{1345}}$ and $\delta_3 = \widehat{P_{1245}, P_{1345}}$. By applying the the Gauss-Bonnet formula, we have

$$\begin{aligned} \cos(\gamma_{1345}) &= \frac{\cos(\delta_1) - \cos(\delta_2)\cos(\delta_3)}{\sin(\delta_2)\sin(\delta_3)} \\ \cos(\gamma_{1245}) &= \frac{\cos(\delta_2) - \cos(\delta_1)\cos(\delta_3)}{\sin(\delta_1)\sin(\delta_3)} \\ \cos(\gamma_{1235}) &= \frac{\cos(\delta_3) - \cos(\delta_1)\cos(\delta_2)}{\sin(\delta_1)\sin(\delta_2)} \end{aligned}$$

Table 7.1: volumes and directions

volumes	Directions
$(T_1 \cap T_2)$	$P_{1235},\ P_{1235},\ P_{1245}$
$(T_1 \cap T_3)$	$P_{1234},\ P_{1235},\ P_{1345}$
$(T_1 \cap T_4)$	$P_{1234},\ P_{1245},\ P_{1345}$
$(T_1 \cap T_5)$	$P_{1235},\ P_{1245},\ P_{1345}$
$(T_2 \cap T_3)$	$P_{1234},\ P_{1235},\ P_{2345}$
$(T_2 \cap T_4)$	$P_{1234},\ P_{1245},\ P_{2345}$
$(T_2 \cap T_5)$	$P_{1235},\ P_{1245},\ P_{2345}$
$(T_3 \cap T_4)$	$P_{1234},\ P_{1345},\ P_{2345}$
$(T_3 \cap T_5)$	$P_{1235},\ P_{1345},\ P_{2345}$
$(T_4 \cap T_5)$	$P_{1245},\ P_{1345},\ P_{2345}$

and,

$$\begin{aligned} vol(T_1 \cap T_5) &= \gamma_{1235} + \gamma_{1245} + \gamma_{1345} - \pi \\ &= -\arccos\left(\frac{\sqrt{3}}{2\sqrt{t^2+3}}\right) + \arccos\left(\frac{t\sqrt{2}}{\sqrt{6t^2+9}}\right) \\ &\quad + \arccos\left(\frac{t\sqrt{2}}{2\sqrt{(2t^2+3)(t^2+3)}}\right) \end{aligned}$$

In a similar way, we obtain

$$\begin{aligned}
vol(T_1 \cap T_2) &= \frac{\pi}{3}\\
vol(T_1 \cap T_3) &= \frac{\pi}{2} + \arccos\left(\frac{t\sqrt{2}}{2\sqrt{2t^2+3}}\right) - \arccos\left(\frac{\sqrt{3}}{\sqrt{2t^2+3}}\right)\\
vol(T_1 \cap T_4) &= \arccos\left(\frac{1}{\sqrt{t^2+3}}\right) - \arccos\left(\frac{\sqrt{3}}{\sqrt{t^2+3}}\right) + \arccos\left(\frac{\sqrt{3}}{3}\right)\\
vol(T_2 \cap T_3) &= \frac{\pi}{2} + \arccos\left(\frac{\sqrt{3}}{6}\right) - \arccos\left(\frac{\sqrt{6}}{3}\right)\\
vol(T_2 \cap T_4) &= \arccos\left(\frac{1}{12}\right) - \arccos\left(\frac{\sqrt{3}}{4}\right) + \arccos\left(\frac{\sqrt{3}}{9}\right)\\
vol(T_2 \cap T_5) &= \arccos\left(\frac{\sqrt{3}}{6}\right) + \arccos\left(\frac{1}{12}\right) - \frac{\pi}{6}\\
vol(T_3 \cap T_4) &= \arccos\left(\frac{t-9}{3\sqrt{5t^2-2t+9}}\right) - \arccos\left(\frac{t-1}{\sqrt{5t^2-2t+9}}\right) + \arccos\left(\frac{1}{3}\right)\\
vol(T_3 \cap T_5) &= \arccos\left(\frac{4t-3}{2\sqrt{5t^2-2t+9}}\right) - \arccos\left(\frac{(4t^2-t+6)\sqrt{2}}{2\sqrt{(5t^2-2t+9)(2t^2+3)}}\right)\\
&\qquad + \arccos\left(\frac{t\sqrt{2}}{\sqrt{6t^2+9}}\right)\\
vol(T_4 \cap T_5) &= -\arccos\left(\frac{t-1}{\sqrt{(5t^2-2t+9)(t^2+3)}}\right) + \arccos\left(\frac{(5t-1}{4\sqrt{5t^2-2t+9}}\right)\\
&\qquad + \arccos\left(\frac{t+3}{4\sqrt{t^2+3}}\right)
\end{aligned}$$

It comes from the Schläfli's formula that,

$$\begin{aligned}
dvol(C_2) &= vol(T_1 \cap T_2)d\theta_{12} + vol(T_1 \cap T_3)d\theta_{13} + vol(T_1 \cap T_4)d\theta_{14} + vol(T_1 \cap T_5)d\theta_{15}\\
&\quad + vol(T_2 \cap T_3)d\theta_{23} + vol(T_2 \cap T_4)d\theta_{24} + vol(T_2 \cap T_5)d\theta_{25} + vol(T_3 \cap T_4)d\theta_{34}\\
&\quad + vol(T_3 \cap T_5)d\theta_{35} + vol(T_4 \cap T_5)d\theta_{45}
\end{aligned}$$

We then multiply $dvol(C_2)$ by 24 and divide it by $\frac{8\pi^2}{3}$ the volume of the hypersphere in $\mathbb{R}^5$ and by 3 (since $\rho = 4$) to obtain the final differential volume $\frac{3}{\pi^2}\int_0^t dvol(C_2)dt$. At $t = 1$, we then derive,

$$\begin{aligned} P_\infty^{IC}((w, w^B), abc/abcd) &= P_\infty^{IC}((w, w^B), abcd - ab) + \frac{3}{\pi^2}\int_0^1 dvol(C_2)dt \\ &= 0.762134 + \frac{3}{\pi^2}\int_0^1 dvol(C_2)dt \\ &= 0.516969 \end{aligned}$$

7.2 Appendix B: A MAPLE sheet for computations

We provide here the maple sheet used to make all of the previous section for the case in detail. The reader can adapt this sheet for all the other case in order to recover our results.

MAPLE SHEET

#Use the linear algebra Maple's library

```
> with(linalg);
```

#Write down the vectors : v0 vector is the unit vector; here, vector v40 is equivalent the parameterized vector v4 of the case in detail and to s6; v4 is equivalent to vector s4 used in the second step. The other vectors, v1, v2 and v3 are derived simply as shown in Appendix A.

```
> v0 := ([1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1]);
> v1 := ([1, 1, 1, 1, 1, 1, -1, -1, -1,-1,-1, -1, 0, 0, 0, 0, 0, 0,0,
0, 0, 0, 0, 0]);
> v2 := ([0, 0, 0, 0, 0, 0, 1, 1, 1, 1, 1,1,-1, -1, -1, -1, -1, -1,0,
0, 0, 0, 0, 0]);
> v3 := ([0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,0,1, 1, 1, 1, 1, 1, -1, -1,-1,
-1, -1, -1]);
> v40:= ([1, 1, 1, 1, 1, 1, -1, -1, -1,-1,-1, -1, 0, 0, 0, 0, 0, 0,t,
t, -t, -t, 0, 0]);
> v4 := ([1, 1, 1, 1, 1, 1, -1, -1, -1,-1,-1, -1, 0, 0, 0, 0, 0, 0,1,
1, -1, -1, 0, 0]);
> v5 := ([0, 0, 0, 0, 0, 0, 1, 1, 1, 1, 1, 1,-1, -1, -1, -1, -1, -1,0,
0, t, t, -t, -t]);
```

#We compute all the angles between two vectors

```
> A12 := angle(-v1,v2);
> A13 := angle(-v1,v3);
> A14 := simplify(angle(-v1,v4));
> A140 := simplify(angle(-v1,v40));
> A15 := simplify(angle(-v1,v5));
> A23 := angle(-v2,v3);
> A24 := simplify(angle(-v2,v4));
> A240 := simplify(angle(-v2,v40));
> A25 := simplify(angle(-v2,v5));
> A34 := simplify(angle(-v3,v4));
> A340 := simplify(angle(-v3,v40));
> A35 := simplify(angle(-v3,v5));
> A45 := simplify(angle(-v4,v5));
```

compute all the differential angles

```
> D12 := simplify(diff(A12,t));
> D13 := simplify(diff(A13,t));
> D14 := simplify(diff(A14,t));
> D140 := simplify(diff(A140,t));
> D15 := simplify(diff(A15,t));
> D23 := simplify(diff(A23,t));
> D24 := simplify(diff(A24,t));
> D240 := simplify(diff(A240,t));
> D25 := simplify(diff(A25,t));
> D34 := simplify(diff(A34,t));
> D340 := simplify(diff(A340,t));
> D35 := simplify(diff(A35,t));
> D45 := simplify(diff(A45,t));
```

STEP 1: we compute the probability of the event “$abcd$ is the collective ranking and $S(B, w_P, \pi, a) > S(B, w_P, \pi, b)$ when d is removed”

Determine all the 19 Vectors ($\vec{v}_6$ to $\vec{v}_{24}$) that form a basis for the orthogonal since vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_{40}$ lie in a 4-dimension space.

```
> N1:=nullspace(matrix([v0, v1, v2,v3 , v40]));
```

Build a super 24x24-matrix made of all the vectors $\vec{v}_0$, $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_3$, $\vec{v}_{40}$ and $\vec{v}_5$ to $\vec{v}_{24}$

```
> MS1 := matrix([v0, V1, V2,V3 , v40,s6,...,s24]);
```

we can calculate the vertexes P_{123}, P_{124}, P_{134} and P_{234}.

```
> f4 := ([0,0,0,0,4*t,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0]);
> P123 := linsolve(MS1,f4);

> f3 := ([0,0,0,24,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0]);
> P124 := linsolve(MS1,f3);

> f2 := ([0,0,12,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0]);
> P134 := linsolve(MS1,f2);

> f1 := ([0,24*t,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0]);
> P234 := linsolve(MS1,f1);
```

Compute the volumes S_{jk}

```
> S12:=simplify(angle(P123, P124));
> S13:=simplify(angle(P123, P134));
> S14:=simplify(angle(P124, P134));
> S23:=simplify(angle(P123, P234));
> S24:=simplify(angle(P124, P234));
> S34:=simplify(angle(P134, P234));
```

#Applying the Schläfli's formula, we the obtain

```
> DV1:=(S12*D12+S13*D13+S14*D140+S23*D23+S24*D240+S34*D340);

> VD1 :=Int(DV1, t=0..1)/(4*Pi^2);
> PF1:=1+24*VD1; #
> Ps:=evalf(PF1); # the fonction "evalf()" gives the numerical value
```

STEP 2: we derive the consistency probability

#Determine all the 18 Vectors (s6 to s24) that form a basis for the orthogonal since vectors V1; V2; V3; V4 and V5 lie in a 5-dimension space.

```
> N:=nullspace(matrix([v0,v1, v2, v3, v4, v5]));
```

#Build a super 24x24-matrix made of all the vectors v0, v1, v2, v3, v4, v5 and s6 to s24.

```
> MS := matrix([v0, v1, v2, V3, v4, v5, v6,...,v_24]);


> #we can calculate the vertexes
> T_1234,T_1235,T_1245,T_1245 and
> T_2345.
> p5 := ([0,0,0,0,0,6*t,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0]);
> P1234 := linsolve(MS,p5);
> p4 := ([0,0,0,0,6,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0]);
> P1235 := linsolve(MS,p4);
> p3 := ([0,0,0,24,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0]);
> P1245 := linsolve(MS,p3);
> p2 := ([0,0,12*t,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0]);
> P1345 := linsolve(MS,p2);
> p1 := ([0,24,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0]);
> P2345 := linsolve(MS,p1);
```

In the following sequence of instructions, we apply the Gauss-Bonnet theorem and then we obtain the volumes $T_{jk} = vol(T_j \cap T_k)$.

```
> Q11 := simplify(angle(P1234,P1235));
> Q12 := simplify(angle(P1234,P1245));
> Q13 := simplify(angle(P1235,P1245));
> C11 := simplify(cos(Q11)- cos(Q12)*cos(Q13))/(sin(Q12)*sin(Q13));
> C12 := simplify(cos(Q12)- cos(Q11)*cos(Q13))/(sin(Q11)*sin(Q13));
> C13 := simplify(cos(Q13)- cos(Q11)*cos(Q12))/(sin(Q11)*sin(Q12));
> T12 := simplify(arccos(C11) +arccos(C12) + arccos(C13) -Pi);

> Q21 := simplify(angle(P1234,P1235));
> Q22 := simplify(angle(P1234,P1345));
> Q23 := simplify(angle(P1235,P1345));
> C21 := simplify(cos(Q21)- cos(Q22)*cos(Q23))/(sin(Q22)*sin(Q23));
> C22 := simplify(cos(Q22)- cos(Q21)*cos(Q23))/(sin(Q21)*sin(Q23));
> C23 := simplify(cos(Q23)- cos(Q21)*cos(Q22))/(sin(Q21)*sin(Q22));
> T13 := simplify(arccos(C21) +arccos(C22) + arccos(C23) -Pi);

> Q31 := simplify(angle(P1234,P1245));
> Q32 := simplify(angle(P1234,P1345));
> Q33 := simplify(angle(P1245,P1345));
> C31 := simplify(cos(Q31)- cos(Q32)*cos(Q33))/(sin(Q32)*sin(Q33));
> C32 := simplify(cos(Q32)- cos(Q31)*cos(Q33))/(sin(Q31)*sin(Q33));
> C33 := simplify(cos(Q33)- cos(Q31)*cos(Q32))/(sin(Q31)*sin(Q32));
> T14 := simplify(arccos(C31) +arccos(C32) + arccos(C33) -Pi);

> Q41 := simplify(angle(P1345,P1235));
> Q42 := simplify(angle(P1345,P1245));
> Q43 := simplify(angle(P1235,P1245));
> C41 := simplify(cos(Q41)- cos(Q42)*cos(Q43))/(sin(Q42)*sin(Q43));
> C42 := simplify(cos(Q42)- cos(Q41)*cos(Q43))/(sin(Q41)*sin(Q43));
> C43 := simplify(cos(Q43)- cos(Q41)*cos(Q42))/(sin(Q41)*sin(Q42));
> T15:=simplify(simplify(arccos(C41)+arccos(C42) + arccos(C43)-Pi));

> Q51 := simplify(angle(P1234,P1235));
```

```
> Q52 := simplify(angle(P1234,P2345));
> Q53 := simplify(angle(P1235,P2345));
> C51 := simplify(cos(Q51)- cos(Q52)*cos(Q53))/(sin(Q52)*sin(Q53));
> C52 := simplify(cos(Q52)- cos(Q51)*cos(Q53))/(sin(Q51)*sin(Q53));
> C53 := simplify(cos(Q53)- cos(Q51)*cos(Q52))/(sin(Q51)*sin(Q52));
> T23 := simplify(arccos(C51) +arccos(C52) + arccos(C53) -Pi);

> Q61 := simplify(angle(P1234,P1245));
> Q62 := simplify(angle(P1234,P2345));
> Q63 := simplify(angle(P1245,P2345));
> C61 := simplify(cos(Q61)- cos(Q62)*cos(Q63))/(sin(Q62)*sin(Q63));
> C62 := simplify(cos(Q62)- cos(Q61)*cos(Q63))/(sin(Q61)*sin(Q63));
> C63 := simplify(cos(Q63)- cos(Q61)*cos(Q62))/(sin(Q61)*sin(Q62));
> T24 := simplify(arccos(C61) +arccos(C62) + arccos(C63) -Pi);

> Q71 := simplify(angle(P1235,P1245));
> Q72 := simplify(angle(P1235,P2345));
> Q73 := simplify(angle(P1245,P2345));
> C71 := simplify(cos(Q71)- cos(Q72)*cos(Q73))/(sin(Q72)*sin(Q73));
> C72 := simplify(cos(Q72)- cos(Q71)*cos(Q73))/(sin(Q71)*sin(Q73));
> C73 := simplify(cos(Q73)- cos(Q71)*cos(Q72))/(sin(Q71)*sin(Q72));
> T25 := simplify(arccos(C71) +arccos(C72) + arccos(C73) -Pi);

> Q81 := simplify(angle(P1234,P1345));
> Q82 := simplify(angle(P1234,P2345));
> Q83 := simplify(angle(P1345,P2345));
> C81 := simplify(cos(Q81)- cos(Q82)*cos(Q83))/(sin(Q82)*sin(Q83));
> C82 := simplify(cos(Q82)- cos(Q81)*cos(Q83))/(sin(Q81)*sin(Q83));
> C83 := simplify(cos(Q83)- cos(Q81)*cos(Q82))/(sin(Q81)*sin(Q82));
> T34 := simplify(arccos(C81) +arccos(C82) + arccos(C83) -Pi);
```

```
> Q91 := simplify(angle(P1235,P1345));
> Q92 := simplify(angle(P1235,P2345));
> Q93 := simplify(angle(P1345,P2345));
> C91 := simplify(cos(Q91)- cos(Q92)*cos(Q93))/(sin(Q92)*sin(Q93));
> C92 := simplify(cos(Q92)- cos(Q91)*cos(Q93))/(sin(Q91)*sin(Q93));
> C93 := simplify(cos(Q93)- cos(Q91)*cos(Q92))/(sin(Q91)*sin(Q92));
> T35 := simplify(arccos(C91) +arccos(C92) + arccos(C93) -Pi);

> Q101 := simplify(angle(P1245,P2345));
> Q102 := simplify(angle(P1345,P1245));
> Q103 := simplify(angle(P1345,P2345));
> C101 := simplify(cos(Q101)-cos(Q102)*cos(Q103))/(sin(Q102)*sin(Q103));
> C102 := simplify(cos(Q102)-cos(Q101)*cos(Q103))/(sin(Q101)*sin(Q103));
> C103 := simplify(cos(Q103)-
> cos(Q101)*cos(Q102))/(sin(Q101)*sin(Q102));
> T45 := simplify(arccos(C101) +arccos(C102) + arccos(C103) -Pi);

#We then derive the final differential volume by apply the Schlafli's formula
> DV :=(D12*T12+D13*T13+D14*T14+D15*T15+D23*T23+D24*T24
> +D25*T25+D34*T34+D35*T35+D45*T45);
#We then derive the final differential volume
> VF := Int(DV, t=0..1)/(8*Pi^2);
#We then obtain our probability
> P:=Ps+24*VF;
> evalf(%);
```

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