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University of Caen

University of Rennes 1

Working Paper

Spatial Inequalities and Expectations:
The Role of Heterogeneity in Krugman's Model

Henri BUSSON

CREM CNRS UMR 6211, University of Rennes 1

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1 Introduction

Spatial inequalities could be very damaging for a country by bringing social tensions [20].
It is a matter of concern for both local and national politicians: national governments has to deal with spatial inequality because of its impact on the welfare of its citizen and local politicians have tried in the past to stop the decline of their city or state. For instance, in Spain, local politicians tried to built a new museum in Bilbao (the Guggenheim museum) in order to make the city more dynamic [10]; or the french department la Charente turn back on its agricol and industrial past by financing a new center in the area of the new images. The most famous exemple of decline of a city is Detroit who was one of the most successful city at the beginning of the 20th century and is now dying because of desindustrialisation, the poor level of qualification of its workforce and racial tensions.
It is relevant to our point of view to take into account the movements of workers with different levels of skills. As pointed out by Glaeser [8] and [9], Houston attracts middle class americans unlike New York who attracts both rich and poor peoples at the same time. In Baccani [2], she

argues that the decision of location are influence by the person's socio economic status. Futhermore, Glaeser and Berry [13] noted that in the US cities, the cities with more human capital tends to attract more skilled workers than the other cities. As a result, an increase in spatial inequality occurs inducing a lack of convergence in income between regions as noted by Barro and Sala-i-Martin [3].

Davezies [20] wrote that with the decreases in public spending that will occur in the next years, inequality will be harder to be fought, and redistribution programs seem not to be a solution? He doesn't believe in making the poorer area more attractive either. He is not necessarily wrong, as noted by Davis and Weinstein [7], there is a remarquable stability of dynamic places troughout history. He said that governements should rather help dynamic cities in poorer areas. There are many advantages to bring more skilled people in a city. Glaeser says in his book [9] that poor areas in cities should be judged by their ability to make poor people less poorer. Indeed, he says that many poors succeed in climbing the social ladder after arriving in a big city. But it is only the case if there are already rich people in the city. In a city like Detroit, people tends to stay poor, there are too few skilled.
In another article with Mare [14], he shows that cities speed the accumulation of human capital. Glaeser and Saiz [15] that only skills city can endure the test of time because they are the only ones that could adapt to economic shocks. They have grown more quickly with city with less human capital because they are becoming more productive. Consequently, cities with not enough human capital are in a poverty trap, making the regional specialization less desirable. On the other hand, too large cities have negative externalities like congestion and environnemental problems (Brueckner [18]).

Regarding literature in NEG, most of its theoretical models are based on the current differential of utility levels : workers move towards city A because the quality of life (modelised by a utility fonction) is higher in that city than in city B. If there are more people in a region A, due to economies of scale, wages will be higher in that region. Hence, more and more people will move toward that region until there will be a full agglomeration. In this context, only history matters here: the biggest region will become automatically the region where everybody ends up.

But if history matters, expectations by individuals could also be an issue as it was formalized by Krugman [22], Matsuyama [19] who introduced expected levels of utility. Indeed it is reasonnable to think that if someone expects the utility levels in the future will be very different from the actuals, he will decide where to go by taking into account this information. Indeed, in some cases, even if region A is the wealthiest, if everybody believes that most people will be in region B in the future, people may move towards region B. Consequently, people make an intertemporal optimization in these models: they choose where to migrate but also when to do it. These models are dynamic.
In those models, workers are considered to be homogeneous. Some papers introduced heterogeneity of workers : Picard and Zeng [23] study thouroughly the importance of the agricultural sector especially when trading agricultural costs are non negligeable and Zeng [6] introduced heterogeneity of preferences. But heterogeneity has been rarely introduced in a dynamic model. The aim of this paper is to introduce heterogeneous qualifications for workers in a dynamic model of NEG a la Krugman [22].

Many theories tried to explain the spatial inequalities (for instance Blanchard and Katz [4] and in particular, Royer [17] argues that history has a major role in the explaination of these inequalities) . The initial conditions, the spatial specialization of the economic activities could explain a lot and they have not been enough studied by economist. By introducing the heterogeneity in a dynamic model with perfect foresight, we can study the impact of history on these inequalities.

But our paper focus more thoroughly on the dispersion of the skills among landscape. Indeed by introducing heterogeneity over workers, skilled ones could be attract in one sector (or in one city) as less skilled ones could be attract in another one, making explicit the spatial dispersion of human capital that does not exist in Krugman's model. We may begin in the basic model only with the difference in wages as the main factor of choice localisation but it could easily be adapted in a more general model with a difference in utility function as showed by Ottaviano, Tabuchi and Thisse [21].

This paper aims at answering several concrete questions:

- Could these policies be efficient and could the path of history be reversed? Is it possible to bring life to a city or a state that is declining?
- Taking into account several levels of skills, will a city with few human capital be able to attract skilled people from other cities and under what conditions? Is it possible to fight spatial inequality in human capital?

In order to answer these questions, it is necessary to reconsider the different economic theories on spatial inequality.

Our methodology is simple: Krugman's model [22] will be reconsidered with two levels of skills. The final equilibriums are to be studied but also how these equilibriums could be reached. Our model deals with movement from one sector to one another but it is easily adaptable to new economic geography. We suppose that skilled workers have an impact on unskilled workers productivity and reciprocally. For instance, an unskilled could have a positive impact on a skilled's productivity by performing simple tasks that allow the skilled to be more focus on the complex ones. And reciprocally, a skilled could increase the unskilled's productivity by being a good manager and helping them to organize in their work. Therefore you have two different kinds of workers and they may not move toward the same location: the skilled could move toward sector X and the unskilled toward sector C. As a result, there are four different final equilibriums instead of two only in Krugman's models [22]: two equilibriums of spatial dispersion of skills appear in the model that are not present in the original model (where the skilled are all located in one sector and the unskilled are all in the other), and two core periphery equilibriums where everybody ends up in the same sector.

Our results are the following:

- In several theoretical papers recensed by Morris and Shin [25], heterogeneity is mainly viewed as a stabilizing force, giving only one determinate equilibrium. For instance, the paper which studied the effect of heterogeneity on a dynamic model was made by Herrendorf, Valentinye and Waldmann [16]. They established that sufficient heterogeneity was ruling out the multiplicity of the equilibrium of the model and the role of expectations. Other papers arrive to the same conclusions with static models (Morris and Shin [25]). The opposite conclusion here is found, where heterogeneity can increase the level of indeterminacy of equilibrium: in this paper a change in expectations leading towards another equilibrium even with a high interest rate and/or high moving costs contrary to Krugman [22]. The intuitive idea is that a new equilibrium path emerges because under heterogeneity the total movement could be divided in small submovements were only one type of workers is moving. As a result, the final conditions of the submovement is different from the final conditions of the global movements. In Fukao and Benabou [11], at the end of the movement there should be no gain for moving towards another area. Therefore some path that were solutions of the equations were not equilibrium paths due to this terminal conditions. But here, at the end of a submovement, it is possible to have some gains because the costs of moving are no longer equals zero and that is enabling the solutions of the equations to become an equilibrium path.

- Furthermore, when equilibriums of dispersion are reachable, a shift in expectations could change the final equilibrium: it could pass from full agglomeration to the dispersion of skills or the reverse. The impact of a type of worker on another type's productivity depends on the substituability of the workers in the congestion process.

In a first section, a litterature review is made. Then, we will present the equations of our theoretical model. After that several intermediate results are enhanced, since these are necessary for achieving final results of this paper that are given and demonstrated in the next section.

2 Literature review

2.1 Krugman's early formalization

Krugman [22] built the following model with two sectors, one labeled C where the productivity per capita is constant and equals to one, and the other labeled X where there are scale economies (due to external economies). There are two possible equilibriums : in one all the workers end up in sector C and in the other all the workers end up in sector X.

Assume that workers want to work in the sector where the salary is the highest. The aim of Krugman's approach is to wonder whether or not the fact that everybody is expected to move into a given sector will make people move into that sector. If the costs of moving to that sector are null, such result will be always true. If the costs are to be positive, then, the decision to move is comparable to an investment decision for workers.

Krugman introduced the concept of a shadow price, which represents the price of being in sector X rather than in sector C: $q(t) = \int_t^{\infty}(w_X - 1)e^{-\rho(s-t)}ds$. w_X represents the actual wage in sector X. It is simply the discounted sum of the difference of wage between the two sector from time t until infinite time. The assets q must be remunered at the world interest rate r and this remuneration must be equal to the rate of capital gains on this asset ($\dot{q}$). The costs of moving from one sector to the other is assumed to be proportional to the number of migrants ($\dot{L_x}$). This second relation means that the marginal gain (the shadow price) of moving must be equal to its marginal costs (which is proportionnal to the number of workers moving; it is a congestion costs).

Therefore, there are two equations, in his model:

$$\dot{q} = rq - w_X + 1$$
$$\dot{L} = \gamma q$$

Krugman demonstrated then that, under certains conditions (economies of scale and costs of moving sufficiently high and interest rate sufficently low) the possible deterministic paths form two spirals diverging from one unstable equilibrium (see figure 1). With that spiral, it is possible

to jump , at a given state of the economy, from a path on that a spiral to another path which leads to the other equilibria. The jump correspond to a different value of the shadow price; it just means the anticipations have just changed (in the shadow price, you have the present and future value of being located in a particular sector). To sum up, q is a jumping variable and it is possible to suddenly go at a different arm of the spirals (which leads to a different final equilibrium) for a given L_x if the anticipations of the future wages change.
The conditions are very intuitive: if the economies of scale are low, there are less interdependance, people are less dependant on other's decisions for their income. If the interest rate is high, it means that future doesn't matter much so people must be less sensitive to future states of the economy. If γ is low, the adjustment between one sector and the other is very slow and the current state of the economy will be near the actual state for a long time.
The change of expectations can occur only in what Krugman calls the overlap.

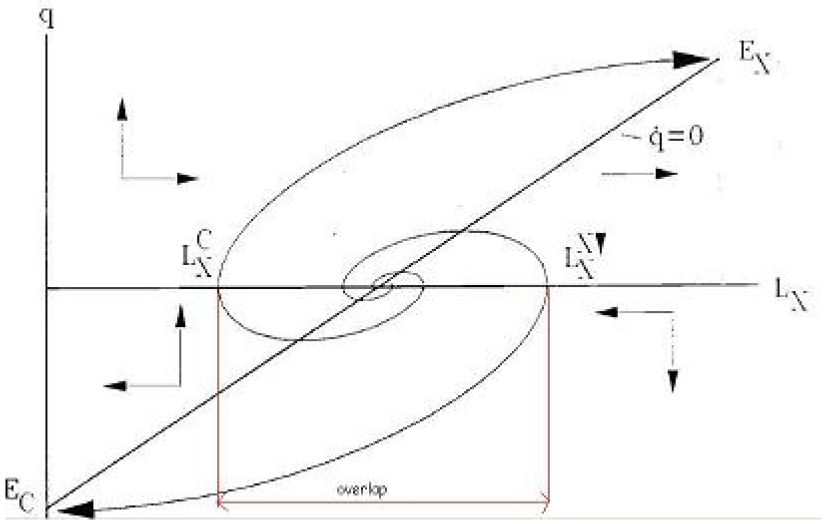

Figure 1: Krugman's overlap from Krugman [22]

The overlap can be rigourously defined as the set of states of the economy for which there are at least two different values of the shadow prices q compatible with an equilibrium path for a given L_X.

If the conditions are not verified, only history matters, the equilibrum paths only make an S-curve. It just means that the sector with the highest number of workers at the beginning is the sector where everybody ends up in equilibrium.

2.2 Extensions of this preliminary work

Matsuyama [19] described the same type of problem in his article but it is a little interest for us because in his model agents have to choose once for all between agricultural sector and manufacturing. We want to apply the model to economic geography problem and people can change of city several times in their lives.

Fukaou and Benabou [11] showed that Krugman was wrong with the equilibrium. First, they say that the workers will all agglomerate in one of the two regions in a finite time T. Therefore, they consider that the expression of q is wrong, being

$$q(t) = \int_t^T (w_X - 1)e^{-\rho(s-t)}ds$$

They asserted that Krugman's potential equilibrium(s) were false and that q has to be equal to zero. If $q \neq 0$ then a worker can wait until everybody moved in order to be able to move without being charged for the congestion costs. They find the same type of spiral paths, but it just ends up at a different value of q. (see figure 2)

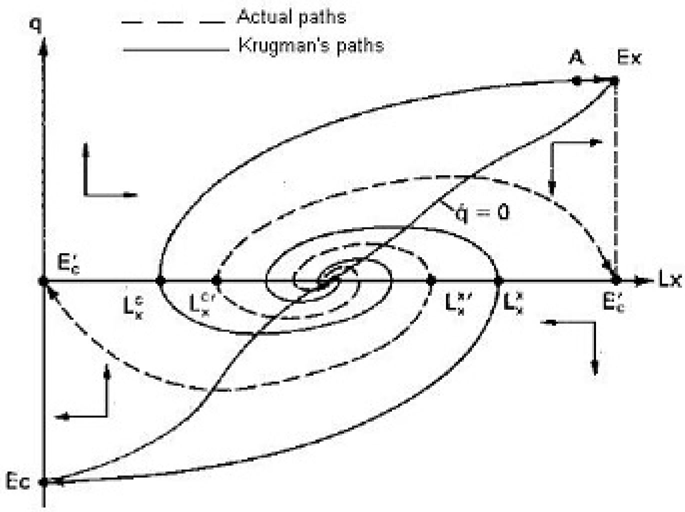

Figure 2: Fukao and Benabou equilibrium paths, from Fukao and Benabou [11]

Ottaviano [12] is the first one that adapted the theory to the new economic geography, Krugman's paper being focused on worker's choice between sector.

Ottaviano, Thisse and Tabuchi [21] revisited new economic geography theory by building a new model with non constant elasticity of substitution and by abandoning Samuelson's iceberg transportation costs. In their paper, they also deal with history versus expectations with their model, finding results that are very similar than Krugman's.

Baldwin [24] studied more rigousrously the difference between traditionnal model and model with forward looking expectations. He found that full agglomeration can occur for any level of trade when there are forward looking expectations; it can only happen for low level of trade with myopia core periphery model.

Wirl and Feichtinger [26] said that economies of scale are not the only factors that could bring expectations to change the course of history. They said that social influence and interactions could make appear such behavior.

Oyama [5] found out that when a state x of the economy verify a certain condition, x is globally accessible when spatial frictions are sufficiently low, it means that you can reach that state from any other state. He also showed that state x is absorbing; it means that for any other state that is close enough to this one, the economy will tend toward that state.

Acemoglu and Robinson [1] adapted the problems to the evolution of social norms through the generations. If everyone have acted by a certain norm in the past, will the future generation act by following this norm or will they evolve by following their expectations?

Herrendorf, Valentinyi and Waldmann [16] built a model starting from Matsuyama's model with heterogeneity in worker's productivity and they showed that heterogeneity was ruling out multiplicity and indeterminancy of equilibrium.

To our knowledge, the Herrendorf and al paper is the only one that introduced heterogeneity into a dynamic model. None paper had studied the dispersion of skills accross the territory.

3 The model

3.1 Definitions and assumptions

There are two sectors in our economy, sector C and sector X and as in previous models: everybody ends up in one sector or the other, all the skills are in one sector and the unskilled are in the other.

Contrary to Krugman, where workers are identical regarding skill level, we assume that two kinds of workers exist, being either skilled or unskilled. We have four equations: two congestion costs equations and two shadow price equations (which depend on the definition of the wages). The shadow price equations indicate us how the shadow price (the opportunity cost of being in one sector X instead of sector C) evolves: if that shadow price becomes positive, then the workers will go toward sector X and if it is becoming negative workers will go toward sector C. The congestion costs equations indicates us how many workers will move from one sector toward another at a date t. They can't all move instantantely because there are congestion costs (the more workers there are the more it costs), they move until the marginal gain of moving equals its marginal costs.

3.1.1 Wage definitions

Considering skill levels, wages are

$$w_{SX} = 1 + \beta_1(L_{sx} - L^*_{sx1}) + \delta_1(L_{ux} - L^*_{ux1}) \quad (1)$$

$$w_{UX} = a + \beta_2(L_{sx} - L^*_{sx2}) + \delta_2(L_{ux} - L^*_{ux2}) \quad (2)$$

Wages are bigger and bigger when more people of the same type comes in the same area. w_{UX} is the wage for the unskilled in the X sector and w_{SX} is its equivalent for the skilled. L_{sx} is the number of skilled people in the X sector and L_{ux} is the number of unskilled in that sector. As it could be assumed that the influence of one type of worker on the productivity of that type is stronger than the influence of the other type, we have $\beta_1 > \delta_1$, $\beta_1 > \beta_2$, $\delta_2 > \beta_2$ and $\delta_2 > \delta_1$. Note that δ_1 and β_2 are not necessarily positive: indeed we could assume that workers have a negative influence on the other type as mentionned in the introduction.

a is the constant productivity level in sector C ($a < 1$). We assume the existence of $L^*_{sx1}, L^*_{sx2}, L^*_{ux1}$ and L^*_{ux2}. But as soon as these equilibrium values exist, there are not necessarily unique.

Our model can also be applied in an agglomeration perspective. Wages are replaced by utility functions.

$$U_s = f(L_s) + g(L_u)$$

U_s is the utility fonction of the skilled, As we have said before, these two equalities can have different implications. If $g' > 0$, it means that the unskilled increases the welfare of the skilled by moving in the same region. It could modelised job complementarity, or it makes one more consumer in the economy. On the contrary, if $g' < 0$, it could modelise the fact that poor people bring with them poverty and crime, or urban congestion.

3.1.2 Shadow price equations

The shadow price becomes respectively for the skilled and for the unskilled

$$q_s(t) = \int_t^{mt} (w_{SX} - 1)e^{-\rho(x-t)}dx \quad (3)$$

$$q_u(t) = \int_t^{mt} (w_{UX} - 1)e^{-\rho(x-t)}dx \quad (4)$$

Notice that the integrals goes from t to mt because its has not been demonstrated yet that the workers are all agglomerated in one region after a finite time T. Therefore, mt belongs to the set $[t, \infty[\cup\infty$. It is exactly the same concept as introduced in the previous section. We simply have two shadow prices instead of one: one per type of worker.

The world interest rate is equal to r and the rate of return of the two q must be equal to this rate:

$$\dot{q_s} = rq_s - w_{SX} + 1 \quad (5)$$

$$\dot{q_u} = rq_u - w_{UX} + a \quad (6)$$

Those two equations are obtained by deriving (3) and (4).

3.1.3 Congestion costs equations

The last equations are obtained by optimizing a production function (nearly the same as in Kurgman's). If the skilled and the unskilled are both moving toward the same sector, the costs of migration are

$$\dot{L_{sx}} + \theta \dot{L_{ux}} = \gamma_1 q_s \quad (7)$$

$$\dot{L_{ux}} + \theta \dot{L_{sx}} = \gamma_2 q_u \quad (8)$$

These equations just pose that the marginal cost of moving is equal to its marginal benefit. The benefit is the shadow price associated with the corresponding level of qualification. The costs are congestion costs. We assume that $\gamma_1 > \gamma_2$: most models assume that unskilled are immobile; we will simply assume that the skilled are more mobile than the unskilled. The costs of migration are made by the migrants themselves; a moving worker generate a negative externality on the others workers, congesting the migration process. The negative externality generated by a skilled on an unskilled (or vice versa) could be different of an externality generated by a skilled on an other skilled : this is why we need the θ coefficient. $\theta \in [0, 1]$.

If the skilled and the unskilled are moving in opposite directions (for instance the skilled are moving in the C sector and the unskilled in the X sector, things are quite different. It depends on the type of externality considered. We will classify the externality in three different types with a parameter μ : $\dot{L_{sx}} = \gamma_1 * q_s - \mu * \theta \dot{L_{ux}}$

- $\mu = 1$ The derivative have opposite signs : the movement of one type of workers are making the movement of the other type less congested. For instance, consider the extension of the model in economics geography, with the choice between two cities with different utility. If the number of houses in one city is fixed on the short term, the number of migrants is congesting the housing market. Therefore, if one person leaves the city, it makes more place for a new migrant.
- $\mu = 0$ The movement of one type of worker has absolutely no impact on the movement of the other type. Think of traffic congestion: the traffic on a highway from one place to another has no impact on the traffic in the other way.
- $\mu = -1$ The externalities are cumulated: even if they are moving in opposite direction, the movement of one type of workers are congesting the movement of the other types. For instance, think of two computers in a network. The downloading from one computer to another could congest the downloading in the opposite direction. In his paper, Baldwin said that the arrival of new people were facilited by the past arrivals. So maybe we could adapt this idea and say that the presence of one type of worker is making easier the arrival of the other type and their departure is making it harder.

Baldwin [24] is the only author who has explained what type of externality it could be. He also mentionned another effect: workers have different subjective cost for migration and the ones for which the costs are the smaller move first. Consequently, migration is becoming harder and harder when the number of workers in a sector, an area becomes smaller. We will consider the three possible different cases.

During the rest of the paper, we will consider only the case with $\theta = 1$ because it is the simplest case where the congestion effects of the skilled and the unskilled are substituable.

As a result the equations become

$$\dot{L_{sx}} + \dot{L_{ux}} = \gamma_1 q_s \quad (9)$$
$$\dot{L_{ux}} + \dot{L_{sx}} = \gamma_2 q_u \quad (10)$$

3.2 Theoretical results

3.2.1 Intermediate results

A few intermediate results are established in this section before presenting the main results of this paper. The movement studied is a complex one and it is necessary to understand several things before the main results.

Signs of q_u and q_s The law of motions are not the same during the entire process, it depends on the signs of q_u and q_s.

3.2.2 q_s and q_u have the same sign

When q_s and q_u have the same sign, if $q_s \neq \frac{\gamma_2}{\gamma_1} q_u$, then one type of workers will have more incentives to move than the other type. If the two are equals, then they will have exactly the same incentive to move.

Proposition If $q_s \neq \frac{\gamma_2}{\gamma_1} q_u$, then only the workers associated with the highest shadow price will move.

Proof Suppose $q_s > \frac{\gamma_2}{\gamma_1} q_u > 0$ and $\dot{L_{ux}} \neq 0$. Then the skilled workers will move according to $\dot{L_{sx}} = \gamma_1 q_s - \dot{L_{ux}}$. But the cost for the unskilled workers become suddenly too strong due to the additional move of a skilled worker $\dot{L_{sx}} + \dot{L_{ux}} = \gamma_1 q_s > \gamma_2 q_u$. The costs for an unskilled worker to move will be higher than his benefits and therefore, L_{ux} has to be smaller.

What is going on is there are more incentive to move for the skilled, and it will end by being too expensive to move for a certain number of unskilled. Therefore, the number of unskilled moving will be reduced, decreasing the cost of moving for the skilled. Consequently, more skilled will move increasing the cost to move for the unskilled and some of them will have to abandon. This pattern will continue until there will be no unskilled. If $\frac{\gamma_2}{\gamma_1} q_u > q_s$ or if q_s and q_u are both negative, the same demonstration is easily adaptable.
As a result, the total movement can be divided in submovements where only one type of workers is moving (ie a submovement is when there is a certain type of worker moving; when the other type begins to move, it is the next submovement).

Proposition The equilibrium path has the same shape than Krugman's. Because only one type of workers will move, for instance if the skilled are moving, L_{us} is a constant and $\dot{L_{us}} = 0$. The equations of motion are almost the same than Krugman's: the matrix is the same, the constant is not the same. The constant doesn't change the roots of the system and we have the same spiral under the same condition $r^2 - 4\beta_1\gamma_1 < 0$.

But if L_{sx} is moving, then q_u is also changing. The skilled have an influence on unskilled wages. How q_u is moving?
It is simply increasing in absolute value until it reaches the value of q_s.

Shadow prices with different signs and $\mu = -1$ If the workers go in the same direction the equations are the same. if they go in an opposite direction the equations become:

$$\dot{L_{sx}} - \dot{L_{ux}} = \gamma_1 q_s \quad (11)$$
$$\dot{L_{ux}} - \dot{L_{sx}} = \gamma_2 q_u \quad (12)$$
$$\dot{q_s} = r q_s - \beta_1(L_{sx} - L^*_{sx1}) - \delta_1(L_{ux} - L^*_{ux1}) + a \quad (13)$$
$$\dot{q_u} = r q_u - \beta_2(L_{sx} - L^*_{sx2}) - \delta_2(L_{ux} - L^*_{ux2}) + 1 \quad (14)$$

As the workers go in opposite directions, the derivatives $\dot{L_{jx}}$ and the shadow prices have opposite signs. Consequently, we have the same case as before, only one type of workers will move, preventing the other type of moving. The same demonstration that we have seen when shadow prices have the same sign could be applied.

Shadow prices with different signs and $\mu = 0$ We can change the equations and say that the movement of a worker who go from sector X to C products no externality for a worker who go from sector C to sector X. In this case, the four equations become

$$\dot{L_{sx}} = \gamma_1 q_s \quad (15)$$
$$\dot{L_{ux}} = \gamma_2 q_u \quad (16)$$
$$\dot{q_s} = rq_s - \beta_1(L_{sx} - L^*_{sx1}) - \delta_1(L_{ux} - L^*_{ux1}) + a \quad (17)$$
$$\dot{q_u} = rq_u - \beta_2(L_{sx} - L^*_{sx2}) - \delta_2(L_{ux} - L^*_{ux2}) + 1 \quad (18)$$

This case is more complex and will not be studied in this paper.

3.2.3 Discontinuity problems

There is a discontinuity of the problem when $q_s = \frac{\gamma_2}{\gamma_1}$. Indeed, the derivatives of L_{sx} and L_{ux} are changing abruptly due to a change in equation. There is a shift from $\dot{L_{sx}} = \gamma_1 q_s$ and $L_{ux} = 0$ to $\dot{L_{ux}} = \gamma_2 q_u$ and $L_{sx} = 0$. The equations are changed for a short interval of time in order to make the system continuous. The continuity is needed in order to have an unique solution of the global system when initial conditions are given.
The intuition is given by the two figures below

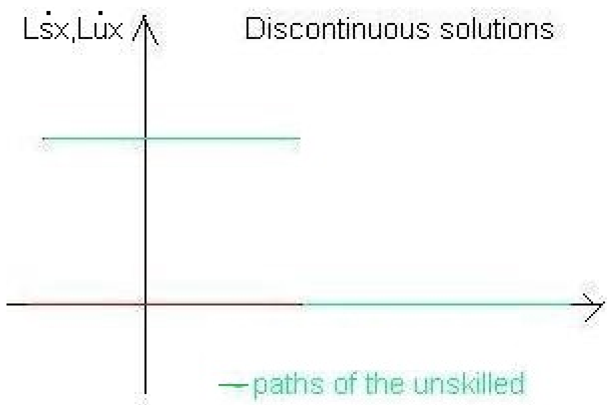

Figure 3: Discontinuous path

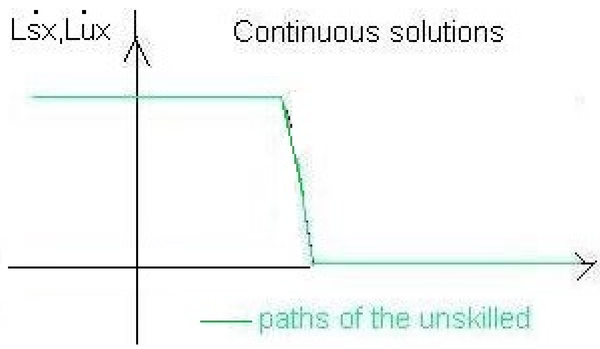

Figure 4: Modified continuous path

Where the discontinuity occurs, the equations of the movement are replaced in order to make a certain transition and to make the derivatives increases (decreases) in a continuous way.

An economic explanation to this phenomena can be given: when the two shadow prices are close enough, the overbidding of one type is not absolutely full. At the end of the transition, the two shadow prices are seperated by ϵ

Proof is given in appendix. Then, the system of differential equations is now continuous, the Cauchy Peano theorem could be applied: for any initial conditions, there exists a solution. As a result, if the terminal conditions of the movement are set at $T = 0$ (the movement occurs when time is negative) then a solution exists. But that solution is not unique because of the ϵ parameter: the solution is unique for a given value of ϵ. But that parameter could varie continuously inside a given interval (given in the proof). And it does not have necessarily the same value for different transitions inside the same movement. For instance, for the first transition, it could be equal to ϵ_1 and for the second transition it could be equal to another value ϵ_2, and for the ith trransition it could be equal to ϵ_i.

3.2.4 Form of the curve when $r^2 - 4\beta\gamma > 0$

In this paragraph, the solution will be studied when $r^2 - 4\beta\gamma > 0$ (this is the condition that Krugman established in order to prevent expectations to play a role in people's choice). For those values, history completely determines the future in the case of homogeneous workers. This will give important preliminary results for understanding the main results.

The worker's movement is described by four equations which forms a linear differential systems. It is easily solvable and the form of the solutions could be described by the following curve in the plane L, q:

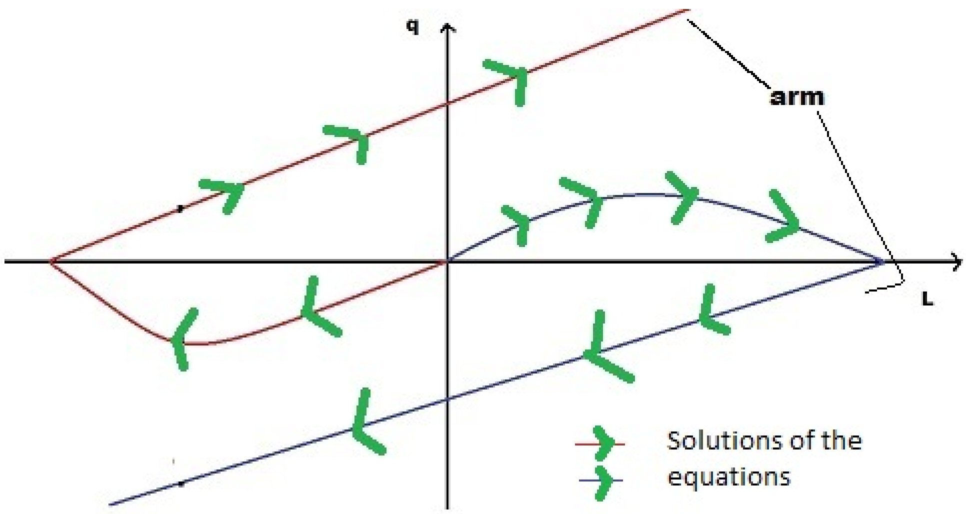

Figure 5: Form of the soluce

The curves are separated in two areas in the rest of the paper :

- the arms: the parts located after the path have crossed the abssis curve (when we follow the arrows on the graph 5). They diverge towards infinity.
- the bells: the parts located before the path have crossed the abssis curve.

But there is a difference between a solution of the equations and an equilibrium path to our economic problems. It is necessary to be a solution of the equations but it is not sufficient because of the terminal conditions that Fukao and Benabou established [11].

Without loss of generality, the case when the skilled are moving is studied. The solution is described by the equations:

$$q_s(t) = Aexp(\lambda_1 t) + B * exp(\lambda_2 t) \quad (19)$$

$$L_{sx}(t) = Cexp(\lambda_1 t) + Dexp(\lambda_2 t) \quad (20)$$

$$q_u(t) = Eexp(\lambda_1 t) + Fexp(\lambda_2 t) + Gexp(rt) \quad (21)$$

λ_1, λ_2, r are the eigenvalues of the systems with $\lambda_1 = \frac{r+\sqrt{\Delta}}{2}, \lambda_2 = \frac{r-\sqrt{\Delta}}{2}$. Three conditions (final or initial) are needed in order to have a unique solution of the system.

The initial conditions are set : $L_{sx}(0) = K, q_s(0) = K2, q_u(0) = K2 - \epsilon$ with $K, K2, K2 - \epsilon$ all positive constants. The solution is uniquely determined with these constraints. The Xcas logiciel (freeware), solved the linear system given in equations (21)-(23) and the following results are obtained:

$$A = \frac{(-K)\beta_1 - K2\lambda_2 + K_2 r}{\lambda_1 - \lambda_2} \quad (22)$$

$$B = \frac{K\beta_1 + K2\lambda_1 - K2r}{\lambda_1 - \lambda_2} \quad (23)$$

$$C = \frac{(K\beta_1\lambda_1 - K\beta_1 r + K2\lambda_1\lambda_2 - K2\lambda_1 r - K2\lambda_2 r + K2r^2)}{(\beta_1\lambda_1 - \beta_1\lambda_2)} \quad (24)$$

$$D = \frac{(-K\beta_1\lambda_2 + K\beta_1 r - K2\lambda_1\lambda_2 + K2\lambda_1 r + K2\lambda_2 r - K2r^2)}{(\beta_1\lambda_1 - \beta_1\lambda_2)} \quad (25)$$

$$E = \frac{(-K\beta_1\beta_2 - K2\beta_2\lambda_2 + K2\beta_2 r)}{(\beta_1\lambda_1 - \beta_1\lambda_2)} \quad (26)$$

$$F = \frac{(K\beta_1\beta_2 + K2\beta_2\lambda_1 - K2\beta_2 r)}{(\beta_1\lambda_1 - \beta_1\lambda_2)} \quad (27)$$

$$G = \frac{(K2\beta_1 - K2\beta_2 - \beta_1\epsilon)}{\beta_1} \quad (28)$$

The problem is whether the movement is located on an arm or on the bell part of the curve. In Krugman's [22], the movement is always located on the bell part. He achieved this result by having one of the final condition at $\dot{q} = 0$ which is only possible outside the arms, on the bell. Fukao and Benabou [11] set the final condition at $q = 0$ which is also possible only outside the arms. But here, the final condition for a submovement is the equality between q_s and $\frac{\gamma_2}{\gamma_1}q_u$; the final condition of the total movement is still $q_s = q_u = 0$. As a result, an equilibrium could temporarily take an arm during a submovement at the condition that it shifts to the bell in oder to be able to reach the terminal condition.
If the movement is located on an arm, then the time when $q_s(t_0) = 0$ should be negative and if not it should be positive because the movement is diverging. By following the arrows on the graph, the path is located on the bell part of the curve before the time $q_s = 0$ and after the movement is located on an arm.

By resolving $q_s(t_0) = 0$,

$$(\lambda_1 - \lambda_2)t_0 = ln(-\frac{B}{A})$$

Therefore, t_0 is negative if and only if $-\frac{B}{A} < 1$.

Remember that $\lambda_1 + \lambda_2 = r$. Three cases arises here

1. B is negative and A is positive.

$$-\frac{B}{A} = \frac{-K\beta_1 + K2\lambda_2}{(-K)\beta_1 + K2\lambda_1} < 1$$

   It is equivalent to

$$K_2(\lambda_1 - \lambda_2) > 0$$

   It is always true as $K_2 > 0$ and $\lambda_1 > \lambda_2$.

2. B is positive and A is negative.

$$-\frac{B}{A} = \frac{K\beta_1 - K2\lambda_2}{K\beta_1 - K2\lambda_1} < 1$$

   It is equivalent to

$$K_2(\lambda_1 - \lambda_2) < 0$$

   It is never true as $K_2 > 0$ and $\lambda_1 > \lambda_2$.

3. Both are negative or positive, then there are no solutions, the curve never crosses the abssis axe.

As a conclusion, the submovement is not on an arm if and only if $-K\beta_1 + K2\lambda_1 < 0$ and $K\beta_1 - K2\lambda_2 > 0$. It is sufficient to have $K\beta_1 > K2\lambda_1$ or $beta_1 > \frac{K2}{K}\lambda_1$.

As q_s is bounded then $\frac{K2}{K}$ is bounded and there exists a constant a_{max} such as $\frac{K2}{K} < \frac{a_{max}}{K}$.

Conseqsuently it is sufficient to have $\beta_1 > \frac{a_{max}}{K}\lambda_1$. Or we could rewrite it $K > \frac{a_{max}}{\beta_1}\lambda_1$.

We could write the same inequality for the skilled, $K_u > \frac{b_{max}}{\delta_2}\lambda_3$ with b_{max} the max of q_u, and K_u the equivalent for the unskilled of K.

We also need to show under which conditions q_s could change its sign and reach an arm:

Notice that $E = \frac{\beta_1}{\beta_2}A$ and $F = \frac{\beta_1}{\beta_2}G$. If the final conditions are reached before the time t_0 where $q_s(t_0) = 0$ then it is impossible to reach the arm. We have

$Aexp(\lambda_1 t_0) + Bexp(\lambda_2 t_0) = 0$. We want to calculate the value of $q_u(t_0)$.

$$Eexp(\lambda_1 t_0) + Fexp(\lambda_2 t_0) + Gexp(rt_0) = \frac{\beta_1}{\beta_2}(Aexp(\lambda_1 t_0) + Bexp(\lambda_2 t_0) + Gexp(rt_0) = Gexp(rt_0)$$

Recall that ϵ is the distance between q_s and q_u at the beginning of a submovement; it has been introduced at the section "Discontinuity problems". For ϵ small enough, and this is possible because ϵ could be taken as small as we want above 0 because it has only a majorant, then G is stricly postive. With the initial conditions taken above, when q_s is equal zero, therefore $q_u > 0$ then the skilled should not be moving no more at the date $t = t_0$ then it is impossible to reach the arm because the final conditions are always met before the shadow price could change its sign. Notice that the shadow price of the unskilled is necesarily positive during all the movements because it is equals to $\frac{\beta_1}{\beta_2} q_s + Gexp(rt)$ and $q_s > 0, G > 0$. Therefore it means that both of the shadow prices have necessarily the same sign during all the movement.

3.3 Equilibriums

In this section we will give and study the final equilibriums. If dispersion is possible, then there are only four final equilibriums that could be stable: where everyone is in sector C or everyone in sector X, where the skilled are in sector C and the unskilled in sector X or the opposite.

3.3.1 Equilibrium of dispersion

A dispersion (ie when all skilled workers will be agglomerated in one sector and all the unskilled will be in the other one) will be possible if and only if the wage for the skilled are higher in the sector where they are located and higher in the other sector for the unskilled with also all the unskilled located in that other sector.

- in the case where the skilled are in sector X and the unskilled in sector C, we have

$$\beta_1(\bar{L_{sx}} - L^*_{sx1}) - \delta_1(L^*_{ux1}) > 0 \quad (29)$$

$$\beta_2(\bar{L_{sx}} - L^*_{sx2}) - \delta_2(L^*_{ux2}) < 0 \quad (30)$$

- where the skilled are in sector C and the unskilled in sector X

$$-\beta_1 L^*_{sx1} + \delta_1(\bar{L_{ux}} - L^*_{ux1}) < 0 \quad (31)$$

$$-\beta_2 L^*_{sx2} + \delta_2(\bar{L_{ux}} - L^*_{ux2}) > 0 \quad (32)$$

If one of the two types of workers will agglomerate in one of the two regions at a finite date T before the other, they won't move again and stay there forever if they are in a stable equilibrium state. For $t > T$, the movement of workers obey to the same equations than in Krugman's [22]. As we've seen before, the roots of the system doesn't change, we only have a different constant. Consequently, all the previous results with one type of workers applies here. Fukao and Benabou [11] showed that, when there is one type of worker, they will all agglomerate in one of the two sector in a finite time and stay there forever.

3.3.2 Final conditions of the movement and stability of the equilibriums

$r^2 - 4\beta\gamma > 0$ The solutions when $r^2 - 4\beta\gamma > 0$ have the shape of a S curve with arms at the end as described before, this is the condition that Krugman [22] established in order to prevent expectations to play a role in people's choice. It is not known for the moment if the boundary is reached in a finite time or not. The Pointcare Bendixson theorem could not be applied here because the dimension of the differential system is four and not two. But as shown in the previous paragraph, the submovement ends always before the shadow price could reach 0. Therefore, there is an inifinite number of submvements leading toward $q_u = q_s = 0$ and the negative value of the shadow prices could never be reached. If it is reached in infinite time, the conditions $q_s = q_u = 0$ is not applied, and the final equilibrium is reached at $\dot{q_s} = \dot{q_u} = 0$ and that equilibrium is stable.

Notice that only the path needs at the end to be on the bell part of the curve in order to be an equilibrium path because the final conditions $q = 0$ or $\dot{q} = 0$ can't be reached on an arm.

3.4 Role of expectations

The expectations are taken into account in $q : \int_t^{mt}(w_{SX} - 1)e^{-\rho t}dt$ is the sum of the discounted difference of the wages in the two sectors from date t to the end of the movement. A change of expectations is possible when, for a given spatial distribution of the economy, we have different value of q that are compatible with the equations. If L_{sx} and L_{ux} are given, the actual difference in wages is also determined and therefore a q of different value means a difference in the futur wages.
As a result, there exists a possibility of a change in anticipations for the skilled if and only if $\exists L_{sx}, L_{ux}$ and $q_s^1(L_{sx}, L_{ux}), q_s^2(L_{sx}, L_{ux})$ such as $q_s^1(L_{sx}, L_{ux}) \neq q_s^2(L_{sx}, L_{ux})$ and q_s^1 and q_s^2 are solutions of the differential equations of motions.

But this definition of change in expectations is incomplete. It doesn't tell us when the change will lead to a change in final equilibrium. Krugman [22] started by solving the differential equations and then showed that when there are different deterministic paths, a change in expectations can occur in order to jump from a path to the other. Each path is leading to a different equilibrium. There are two different paths if and only if $r^2 - 4\beta\gamma < 0$. q become a jumping variable which will be able to go from an equilibrium path to another, depending on the expectations of the workers.

3.4.1 Expectations when both shadow prices have the same sign

In this section, only the equations where both the shadow prices have the same sign will be studied.

A change in expectations is possible even if $r^2 - 4\beta\gamma > 0$ Suppose L_{sx} is such that there are more workers in the C sector and $r^2 - 4\beta\gamma > 0$. In these conditions, Krugman results [22] say that the final equilibrium is at $\bar{L_s}$. But with the heterogeneity of workers, an equilibrium path with these initial conditions can lead toward an equilibrium where everobody is in the X sector. A positive shadow price with a low value of L was not possible in Krugman's model because the terminal condition needs to be $q = 0$ and that is only possible on the bell part of the curve. With the heterogeneity, the terminal condition of a submovement changes and it becomes $q_s = q_u$. Therefore if we are located on an arm of the curve, we need to find a time t_1 for which $q_s(t_1) = q_u(t_1)$ with $t_1 > 0$. And after that, as the final condition of the total movement is $q_s = q_u = 0$, as the next submovement starts, it is necessary to be located on the bell part of the curve.

The rigourous demonstration is in the appendix. We will give the intuition here.

Firstly, if the state of the economy is on an arm, there is necessarily a time t where $q_s(t) = q_u(t)$. At this point, a shift could occur: as the terminal condition could only be reached if the economy is located on a bell, it is possible that during the submovement of the unskilled, the ratio $\frac{q}{L}$ become low enough for the skilled to be located on the bell part of the curve when the skilled are moving again. Therefore when the unskilled are moving, the economy is on an arm for the skilled at the beginning of the movement and it is on the bell at the end of the movement. As a result, when the skilled are moving again they are located on the bell part and terminal condition could be met there. The arm becomes therefore a possible part of an equilibrium path and it is precisely that part of the curve that enables the less populated sector at the beginning to become the more populated at equilibrium.

Suppose the economy is located on an arm for the skilled and on a bell for the unskilled. Remember that to be located on an arm we need to have a ratio $\frac{q}{L}$ sufficiently high (the pink area on the graph) and a ratio $\frac{q}{L}$ low enough to be located on the bell (the grey area on the graph).

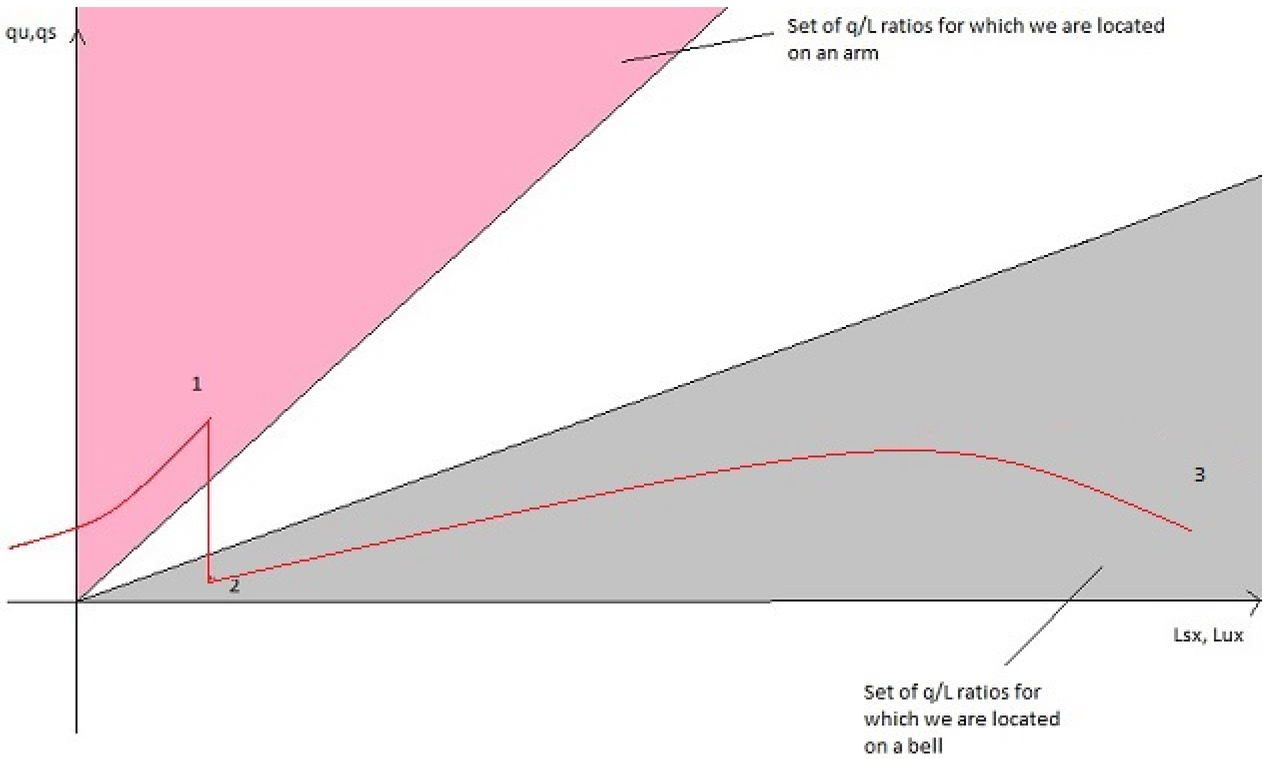

Figure 6: New equilibrium path

On the figure 3.4.1, the path taken by the shadow price of the unskilled is in red. It starts on an arm, and it is located at the point 1 on the graph. The movement is divided in three steps. At the point 1 the unskilled are moving. They are located on the bell. As the skilled are not moving, their shadow prices are only moving vertically in the space q_s, L_{sx}. At the end of

the submovement, the skilled are located on the bell part of the curve. That shift is plausible because in the q_u, L_{ux} plane, the economy is located on a bell.
In the rigourous form of the demonstration, a sufficient condition is found for the unskilled to stay on the bell part of the curve: the increasing returns of the unskilled must be sufficiently strong. Intuitively, because the unskilled are all agglomerated in one area and because they have sufficiently large increasing returns, this make the interdependance of their decisions more important, that the change of expectations is possible. It is these increasing returns that helps the shadow price of the skilled to become sufficiently low to be located on the bell part of the curve. Therefore, as q_u and therefore q_s could not be stronger than a certain value, and because L_{sx} is increasing again and again over time because q_s stays positive, the ratio should become low enough to be at a certain moment on the bell part of the curve.

To finish, it is shown that once both of the two submovements are located on the bell part of the curve, the terminal conditions could be reached.

In other words the paths are the following (see figure 3.4.1):

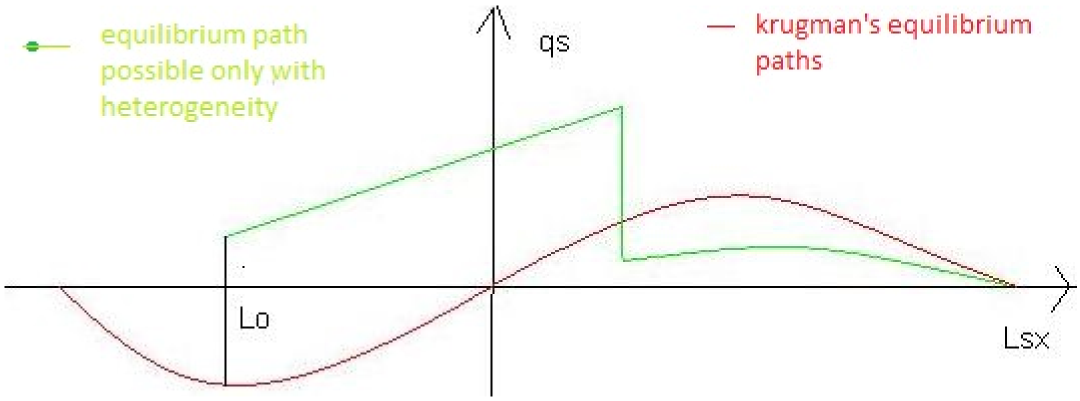

Figure 7: Possible Paths

At the point L_0, you have two different equilibrium paths instead of one in Krugman's case leading toward two different equilibriums.

Lemma The set of the states of the economy for which wages are equalized for one type of worker is linear and of one dimension. The existence of a couple L^*_{sx1}, L^*_{ux1} has been assumed in this model for which the wages are equalized for the skilled. Suppose there is another spatial configuration L'_{sx1}, L'_{ux1} for which the wages are equalized. Therefore:

$$w_{SX} - 1 = \beta_1(L'_{sx} - L^*_{sx1}) + \delta_1(L'_{ux} - L^*_{ux1}) = 0 \tag{33}$$
$$\tag{34}$$

We can write $L'_{sx} - L^*_{sx1} = \Delta_1$ and $L'_{ux} - L^*_{ux1} = \Delta_2$. By replacing, we have $\beta_1(\Delta_1) + \delta_1(\Delta_2) = 0$ and $\Delta_1 = -\frac{\delta_1}{\beta_1}\Delta_2$. This gives

$L'_{sx} - L^*_{sx1} = -\frac{\delta_1}{\beta_1}\Delta_2 = -\frac{\delta_1}{\beta_1}(L'_{ux} - L^*_{ux1})$. This is a linear relation between L'_{sx} and L'_{ux}. We could applied exactly the same demonstration for the unskilled. Then, we have two segments that describe the spatial configurations for which the wages of one type of worker are equalized between the sectors.

We define the subset L_0 of the spatial configurations : L_0 is all the couples L_{sx}, L_{ux} that are either over both the values $\frac{a_{max}}{\beta_1}\lambda_1$, $\frac{b_{max}}{\delta_1}\lambda_3$.

3.4.2 L_{sx} and L_{ux} large enough in order to be located only on the bell part of the curve

Sufficient condition for history to determine completely the path If both types are located on the arms, then it is impossible for the expectations to play a role. Examine the movement of one type of the workers. During a submovement where this type moves, it is impossible to reach by itself the bell part of the curve by definition. During a submovement where the other type is moving, the labor $L_{ix}, i = s, u$ remains contant and q_i is necessarily increasing (a quick look t the equations and notive that it is the linear combinaison of q_j and e^{rt} which are both increasing). Then the ratio $\frac{q_i}{L_{ix}}$ is increasing; therefore it stays on an arm.

Role of δ_1 and β_2 In the previous paragraph a sufficient condition for expectations to play a role has been seen. But is it necessary? The influence of a type of workers during a submovement of the other type will be studied. In the previous paragraph, only the influence of a type of worker on worker of the same type has been studied and neither δ_1 nor β_2 appeared in the previous sections.

Without loss of generality, the impact of the unskilled on the skilled only will be studied and what could be the influence of δ_1. Two cases are possible :

- During a submovement of the skilled. As L_{ux} is a constant, it has absolutely no impact on the eigenvalues and on the shape of the submovement.
- During a submovement of the unskilled, only the path of the unskilled has been studied previously. What happened of the path for the skilled ? It is necessary to look at the eigenvalues of the matrix of the differential system.

  The matrix is $\begin{matrix} 0 & \gamma_2 & 0 \\ -\delta_2 & r & 0 \\ -\delta_1 & 0 & r \end{matrix}$ L_{sx} is a constant, consequently there are only 3 equations with the three variables L_{ux}, q_u, q_s.
  The characteristic polynom is $X^3 - 2X^2 + X(r^2 - \delta_2\delta_2) + \gamma_2\delta_2 r$.
  The eigenvalues are r, $\frac{r+\sqrt{r^2-4\delta_2\gamma_2}}{2}$, $\frac{r-\sqrt{r^2-4\delta_2\gamma_2}}{2}$. It could be noticed that δ_1 is not present in the eigenvalues therefore it has no role in the equilibrium paths. The reason why is because the cofacteur where it is supposed to appear is to be multiplied by zero, because q_s has absolutely no influence on the derivatives of L_{ux}.
  Therefore, δ_1 plays no role in the shape of the equilibrium path. The eigenvalues depend only on the variable r, δ_2 and γ_2.

δ_1 is only a constant in the equations, therefore, it plays no role in the shape of the submovement.

3.4.3 Expectations with different signs when $\theta = 1$ and $\mu = 1$

We need some intermediate results before studying the expectations.
The costs of moving are no longer a problem because the skilled workers are the unskilled are moving in opposite directions. Consequently, the derivative $\dot{L_{sx}}$ and $\dot{L_{ux}}$ have opposite sign. When one unskilled worker is moving, he is reducing the cost of moving for the skilled worker for instance. As a result, when one more skilled worker is moving, he is reducing the cost of migration for the unskilled. So all the migration occurs almost instantaneously. We assume that for the cost to be negligeable, if there is one type of worker moving, a worker of the other type must be moving in the opposite direction. The question is for how long the costs are no longer a problem: when q_s or q_u is changing its sign or when all of the workers of one type are agglomerated in one region. When that happens, this type of workers i will not move and therefore $\dot{L_{ix}} = 0$ and for the other type j we have $\dot{L_{jx}} = \gamma_k * q_j$.

But another problem arises. Fukao and Benabou [1993] showed that, on a equilibrium path, there must be no arbitrage possiblities: the workers can have no gain by waiting to move to the other sector at a different time. Here, we take the perspective of the type of workers that has the higher q when the two types has q with the same sign (assume that type is the skilled, the same demonstration can be applied with the unskilled) at a certain date t_0. We have consequently $|q_s| > \frac{\gamma_2}{\gamma_1}|q_u| > 0$. At a certain date $t > t_0$, assume that $q_u = 0$ with the derivative of q_u telling us that it will change its sign. Therefore, the two type will have q with different sign and the costs will be negligeable. Fukao and Benabou showed that, if $q > 0$ and if the costs of moving become suddenly equals to zero, this can't be an equilibrium path. Therefore, if $q_u = 0$ at t then $q_s = 0$ at the same date. The only question that remains is that whether the derivative of q_s and q_u has the same sign at t.
The derivatives $\dot{q_s}$ and $\dot{q_u}$ are continuous as a linear combination of continuous function (L_{sx}, L_{ux}, q_s and q_u are all continuous because solutions of the differential system). Therefore, admit that the two has the same sign at a date t. If one of the q equals 0 at t_1, the other one should also equals zero and the derivatives have the same sign when the two are equal to zero. As a result, q_s and q_u will change their sign simultaneously because the derivatives are continuous and if they have the same sign at a date t_1, they will have the same sign for $t \in [t_1, t_1 + e]$ with e small enough.
The main conclusion of this paragraph is that, when the two q_s and q_u starts with the same sign, they keep the same sign at least for as long as one of the two types of workers will stop moving. Because if they move in the opposite direction, costs will become null and a worker could make a gain by deviating from the equilibrium path and move when these costs will become null.

Expectations complements At every moment, workers can anticipate a change of equilibrium towards a dispersion one instead of a full agglomeration one if the dispersion equilibrium is possible. Because the costs become null, the equilibrium occurs instantanely. This change of expectation can occur for every distribution between the two sectors.

Theorem Suppose dispersion is impossible. Full agglomeration in the sector with economies of scale is an absorbing state (ie $L \in [0, 1]$ is absorbing if there exists $\epsilon > 0$ such that any equilibrium path from any $x \in B_\epsilon(x)$ converges to x).

We will adapt here the work of Oyama [5]. We note $F_s(L_{sx}) = \frac{df_s}{dL_s}(L_s)$ with $f_s(L)$ the difference in the actual wages between sector C and sector X for the skilled. The analogue function for the unskilled is noted f_u. We define the function $H(L,q)$. As dispersion is impossible, the two shadow prices has necessary the same sign during all the movements.

$$H(L,q) \quad = \quad F_s(L_s(t)) + F_u(L_u(t)) + 1_{SX}\gamma_1\frac{q_s^2}{2} + 1_{UX}\gamma_2\frac{q_u^2}{2} \tag{35}$$

Let (L(), q()) be a solution of the equations (9)-(12). Then, $\forall t > 0$

$$\frac{d}{dt}H(L,q) \geq 0 \tag{36}$$

with equality holding if and only if $q_s(t) = 0$ on 1_{SX} and $q_u(t) = 0$ on 1_{UX}.
Proof

$$\frac{d}{dt}H(L,q) = f_s(L_s)\dot{L_s} + f_u(L_u)\dot{L_u} + \gamma_1 1_{SX}\dot{q_s}q_s + \gamma_2 1_{UX}\dot{q_u}q_u$$

$$\frac{d}{dt}H(L,q) = 1_{SX}(f_s(L_s) + \dot{(q_s)})\gamma_1 q_s + 1_{UX}(f_u(L_u) + \dot{(q_u)})\gamma_2 q_u$$

$$\frac{d}{dt}H(L,q) = 1_{SX}r\gamma_1 q_s^2 + 1_{UX}r\gamma_2 q_u^2$$

It is always positive or nul and we obtain easily that it is nul if and only if $q_s(t) = 0$ on 1_{SX} and $q_u(t) = 0$ on 1_{UX}.

The rest of the proof is almost the same : Oyama used the Lemma 5.6

Let L be an equilibrium path from L_0. If $\hat{L}$ is an accumulation point of x, then
(1)$F(\hat{L}) \geq F(L_0)$, and
(2) $\hat{L}$ is an equilibrium state.

Proof t_k is a sequence such that $lim(t_k) = \infty$ and $limL(t_k) = \hat{L}$
We also define $L*(t), q*(t)$ such as $L*(t) = lim_{t\to\infty}L(t_k+t)$ and $q*(t) = lim_{t\to\infty}q(t_k+t)$

Oyama [5] showed that $\frac{d}{dt}H(L*(t), q*(t)) = 0$. The same demonstration could also be applied here. Hence we have $q_s(t) = 0$ on 1_{SX} and $q_u(t) = 0$ on 1_{UX}. Therefore, $\dot{L_s(t)} = 0$ when the skilled are supposed to move and $\dot{L_u(t)} = 0$ when the unskilled are supposed to move anymore. Consequently, we could say that nobody is moving and $\hat{L}$ is an equilibrium state.

We have one more possible option: that workers are moving during the transition state. One type of worker alone can't move. Therefore, either both types of workers are moving or we have $q_s = q_u = 0$ and nobody is moving. But if we have a transition state , at the end, only one type of workers is supposed to move, we can't have another transition state and we can't have nobody moving by definition at the end of that transition state (at the end of the transition state, it is built in order to have one shadow price greater than the other).

The rest of the demonstration is exactly the same as in Oyama [5] and therefore we just showed that the maximizer of the potential is an absorbing equilibrium state.

3.4.4 Expectations with different signs when $\mu = -1$ or $\mu = 0$

The differences with case where $\mu = 1$ are workers don't necessarily go in the same direction once the movement has begun.

The results on the final equilibrium doesn't change. Because when the shadow price fo the type of the workers is moving become null, the other shadow price is necessarily superior in absolute value therefore there are an infinite number of submovements leading toward equilibrium and in none the final conditions are actually reached.
The demonstration of a change of expectations with high interest rate could be applied here.

The case $\mu = 0$ will not be studied here because it seems rather complex.

4 Discussion

This paper aimed at knowing that if spatial inequalities could be entirely explained by history or if expectations could play a role. The method consisted in introducing heterogeneity in Krugman's model [22]. Introducing this enables four different equilibriums states instead of two only. It has been established that it is possible for expectations to play a role even with high interest rates if the increasing returns are strong enough. The explaination of this result lies in the difference in the terminal condition. In Krugman's model [22], that condition as to be $q = 0$ but with heterogeneity, the total movement could be subdivided in a large number of small submovements where only one type of workers is moving. In each of these submovements there is a terminal condition different than $q = 0$ and that enables to take a new equilibrium path. As a result, heterogeneity is a factor of instability and multiplicity. Another results is that a theorem established by Oyama [5] has been readapted to this paper in a particular case when $\mu = 1$. Consequently, the sector with economies of scale is an absorbing state.

Our model could be extended to new economic geography : instead of wages functions, more general utility functions could be employed. The results could hold if and only if the utility differiential between region A and region B is linear in the number of workers in region A. In the non linear case, it could still hold if it is locally linear. Remember that the all movement is divided in small submovements. Therefore, if it is linear during the restrictions of a submovement, whatever that submovement is, the results hold.

A Discontinuity problem

In the case where $q_s = \frac{\gamma_2}{\gamma_1} q_u$, the problem of discontinuity arises. Indeed, one type of worker suddenly stops moving and another begin to move with a derivative instantanely equals to $\gamma_j q_k$. The continuity of the system is needed in order to have the necessary hypothesis for the Cauchy Peano theorem. Without loss of generality, consider the case where the skilled workers are moving, and then suddenly the unskilled are moving (the opposite case require almost the same discussion). An additinal assumption is made : when $\frac{\gamma_2}{\gamma_1}|q_u| - \epsilon_c \geq |q_s| \geq \frac{\gamma_2}{\gamma_1}|q_u| + \epsilon_c$, with ϵ_c very small, the law of motion is a little different.

L_{sx}^{eq} and L_{ux}^{eq} are the values such as $q_s(L_{sx}^{eq}, L_{ux}^{eq}) = \frac{\gamma_2}{\gamma_1} q_u(L_{sx}^{eq}, L_{ux}^{eq})$. As the unskilled workers are moving, $\frac{\gamma_2}{\gamma_1} \dot{q_u}(L_{sx}^{eq}, L_{ux}^{eq}) > \dot{q_s}(L_{sx}^{eq}, L_{ux}^{eq})$. As the derivatives are continuous, there exists ϵ_v such as ,for every spatial distribution x, $||(L_{sx}^{eq}, L_{ux}^{eq}) - x|| \leq \epsilon_v => \frac{\gamma_2}{\gamma_1}\dot{q_u}(x) > \dot{q_s}(x)$. An area has just be defined where the derivative of the shadow price of the unskilled is greater than the derivative of the shadow price of the skilled.

The derivatives of the shadow prices are bounded, there exists m_u, M_u, m_s and M_s such as $m_u \leq \dot{q_u} \leq M_u$ and $m_s \leq \dot{q_s} \leq M_s$. ϵ_c is such as $Max||q_s(x)|-\frac{\gamma_2}{\gamma_1}|q_u(x)||$for $x \in B((L_{sx}^{eq}, L_{ux}^{eq}), \epsilon_v)$. We want to define an economic transition from the skilled movement toward the unskilled movement. At the end of this transition, it is necessary to have $\frac{\gamma_2}{\gamma_1}q_u(x) > q_s(x)$ because the law of the movement is changing for a short period and the fact that it is the unskilled that are supposed to start moving. We note Δ_t the lenght of the transition. Suppose the transition starts within the ball $B((L_{sx}^{eq}, L_{ux}^{eq}), \epsilon_v)$.

It is necessary to have $\gamma_2 q_u(t)+min(\dot{q_u})\Delta_t > \gamma_1 q_s(t)+max(\dot{q_s})\Delta_t$ where t indicates the time of the beginning of the transition. If that condition is respected, we will have $q_u(t+\Delta_t) > q_u(t+\Delta_t)$. It is sufficient to have $(M_s - \frac{\gamma_2}{\gamma_1}m_u)\Delta_t < \epsilon_c$.

That condition could be rewritten $\Delta_t < \frac{\epsilon_c}{(M_s-\frac{\gamma_2}{\gamma_1}m_u)}$. We want to keep an autonomous system of equation. Therefore, we need to define the duration of the transition for a given interval of state and not for a given interval of time.

We note $\dot{q_s^M}$ the maximum of the derivative on the time interval Δ_t. We have $\Delta_t \dot{q_M} = 2\epsilon$ with 2ϵ the size of the interval of states of the transition. Therefore, we need $\epsilon \leq \frac{\dot{q_M}\epsilon_c}{2(M_s-\frac{\gamma_2}{\gamma_1}m_u}$. It is satisfied if $\dot{q_M} < Min(m_u, m_s)$.

We define λ as a function of q_s such as $\lambda(q_s) = 1$ when $|q_s| - \frac{\gamma_2}{\gamma_1}|q_u| = \epsilon$ and $\lambda(q_s) = 0$ when $\frac{\gamma_2}{\gamma_1}|q_u| - |q_s| = \epsilon$. λ is stricly increasing and continuous, it is not function of the time but a function of the difference between L_{sx} and L_{ux}. The movement of the workers when $|q_s| \in [\frac{\gamma_2}{\gamma_1}|q_u| - \epsilon, \frac{\gamma_2}{\gamma_1}|q_u| + \epsilon]$ is as follow :$\lambda\frac{\dot{L_{sx}}}{\gamma_1} + (1-\lambda)\frac{\dot{L_{ux}}}{\gamma_2} = max(q_s, q_u)$. When $\lambda = 1$, the derivative of q_u is stronger than q_s's, by the construction of ϵ we made above, and $\frac{\gamma_2}{\gamma_1}|q_u| > |q_s|$ therefore only the unskilled are moving at this moment. It is easy to verify that both of our movements are now continuous.

This is mathematical trick to have a continuous movement, but we can give also an economics explanation to this behaviour. When the gains from moving to another sector are very close, it is unlikely that only one type of worker will move. When the gain are very similar, we simply suppose, that the incentive of moving for the two types of worker are almost the same and therefore the movement will be a mix of the two types moving. We saw earlier that if the gain of moving are stronger for the skilled for instance, another skilled worker might be able to move and reduce the incentive of the unskilled to move. But suppose the gain for the skilled to move are very small, the perception of the workers may be what matters and they might necessarily see it clearly that they have more incentives to move.
The Cauchy Peano theorem gives only the existence of a local solution. The interval of time of its existence must be bounded. As a result, it is possible to take $t = 0$ when an equilibrium is reached and to go backward in time until a time T_{ini} negative sufficiently high in aboslute value.

B Change of expectations with $r^2 - 4\beta\gamma > 0$

Suppose $t = 0$ at a moment where the skilled just starts moving; the same notation as in the subsection (4.3) are taken and $K < 0$, $K_2 > 0$. Therefore q_s is located on an arm in the skilled movement. Is there a solution for which $q_s(t_1) = q_u(t_1)$? it is obvious because when $t \to \infty$, then q_u is greater than q_s because in $+\infty$, q_u is equivalent to $Gexp(rt)$ which is greater than the equivalent of q_s (equals to $Aexp(\lambda_1 t)$) as $r > \lambda_1$.
Suppose at the end of the submovement that L_{sx}, L_{ux} are both over the segment that equalize

the wages.
Suppose that we are on the bell part of the curve for the unskilled but we are on an arm for the skilled. In order to be located on the bell part of the curve, it is necessary and sufficient to have, as it has been shown previously, that $L_{ux}\delta_2 > q_u(0)\lambda_3$ and to be located on an arm we need to have $L_{sx}\beta_1 < q_s(0)\lambda_2$. We take L_{sx} very close to zero and $q_s(0)$ sufficiently high; we take $q_u(0)$ and $q_s(0)$ close enough but q_u is greater than q_s.

The demonstration will be in two steps. Firstly, as it will be shown with these initial conditions this could lead to a final equilibrium where everyone ends up in sector X by respecting the equations and the final global conditions of $q_s = q_u = 0$ at the end of the total movement. Therefore we are about to describe an equilibrium path.
Secondly an initial negative L_{sx} could lead to this path, therefore allowing a change in expectations.

The unskilled are beginning to move. It is necessary to find a condition that when the skilled are moving, the unskilled stay on the bell part of the curve. Indeed, if the unskilled are located on the bell part of the curve, it will stay on that part during a submovement of the unskilled. It may quit that part during a submovement of the skilled only. Define $f > 0$ such as $\frac{q_u(0)}{L_{ux}(0)} = \frac{\delta_2}{\lambda_3} - f$ $t = 0$ at the beginning of a movement of the skilled and we would like to have $\frac{q_u(t)}{L_{ux}(t)} \leq \frac{\delta_2}{\lambda_3} \forall t$.

A first lemma is needed.

Lemma A minimal value of ϵ can be considered for a finite number of transition, by starting at the initial conditions mentionned above. Suppose the minimal value is below the maximal value defined at the paragraph "discontinuity problems". The only problem is the positivity of the coefficient $G = K2 - \epsilon$. But here, it is known that $K2$ has a minimal value as there is a finite number of transitions and the shadow price is never equals to zero because the shadow price of the other type is always met before as shown previously.

As L_{ux} is constant during a submovement of the skilled, it is sufficient that $q_u(t)$ to be lower than a certain constant R during all the submovements of the skilled. As we have taken a finite number, L_{ux} is bounded, there exists $MinLu$ such as $L_{ux} > MinLu$ and therefore $\frac{q_u}{L_{ux}} < \frac{q_u}{MinLu}$.

As a result, if q_u inferior to a constant for all the submovements then $\frac{q_u}{MinLu} < \frac{\delta_2}{\lambda_3}$. Suppose the maximum of all the submovements has been reached at time t_2. It has been established that $q_u(t_2) = \frac{(-K)\beta_1 - K2\lambda_4 + K_2 r}{\lambda_3 - \lambda_4})exp(\lambda_3 t_2) + (\frac{K\beta_1 + K2\lambda_3 - K2r}{\lambda_3 - \lambda_4} exp(\lambda_4 t_2)$ with $K, K2$ tha values of L, q at the beginning of the submovement where the maximum has been found. Therefore, there exists two positive constants $R1, R_2$, $q_u(t_2) = -K\beta_1 R1 + K_2 R_2$.
Consequently, $q_u(t_2) < R$ implies that $\beta_1 > \frac{K_2 R_2 - R}{K R_1}$. It is easy to verify that $q_s - q_u$ is bounded and therefore one can conclude that if R is chosen low enough then q_s will never reach its maximal value (and that would be impossible to continue to verify the equations and to have a q has a maximal value (the derivative would likely to be positive , therefore mathematically q_s would have to increase and it would have been impossible economically).
The condition is possible for a sufficiently large value of β_1. In order to keep the condition $r^2 - 4\beta_1\gamma_1 > 0$ true, the value of β_1 could be increased as the value of γ_1 decreased as there is no condition on that last parameter.

Consider that condition satisfied. The submovement is necessarily converging. Because the skilled are agglomerating more and more in the sector X. As there is a minimal value of ϵ, there is a minimal fraction of the workers that are moving towards sector X. Therefore L_{sx} is increasing and if the number of submovements is large enough, it should be able to reach any value before full agglomeration since there is a minimal fraction of the workers noted Mfw (one could take a number of submovements equals to $\frac{L_{sd}}{Mfw}.q_s$ is bounded as shown in the previous paragraph. Therefore, at a certain moment $\frac{q_s}{L_{sx}}$ should become small enough to be located on the bell part of the curve. As a result, the two types of workers are located on the bell part of the curve and the final conditions of all the movements can be reached.

Once both the two types of workers are on the bell part of the curve, the minimal value of ϵ is given up, it must be possible to take it as small as one wants.

But the demonstration is not finished yet. It is necessary to prove that the final conditions established by Fukao and Benabou [11] are met. For L suffiently large, the derivative of q is negative. Therefore, if both of the type of workers are located on the bell part of the curve, their equilibrium path will stay on it forever if both L's are sufficiently high enough to make the derivatives of the q's negative. And if ϵ is taken small enough, then both q_u and q_s are becoming small and smaller. As it is decreasing and bounded, it is converging. Suppose that there are converging towards a value q_{mn} that is not equals to zero. Then it is possible to take a q at the beginning of a submovement such as $q - q_{mn} = \frac{\epsilon_2}{2}$ with ϵ_2 as small as we want. Suppose the movement ends, the value of the lenght next transition could be changed. Suppose we increase ϵ such as the minimal decreases of the shadow value is greater than $\frac{\epsilon_2}{2}$.

Now, return to the initial conditions with L_{sx} very small, the skilled on the arms and the unskilled on the bell part of the curve. The second step of the demonstration implies that an initial negative L_{sx} could lead to this equilibrium path.
The unskilled were moving at time $t = 0$ with these conditions, it means that there was a transition if we are going back in time. The minimal lenght of the transition is Δ_t, defined in the previous section. Therefore, we simply have to take L_{sx} small enough in the final conditions of that submovement in order to be sure that the initial conditions are for L_{sx} negative. Because there is a miniaml non null value for $\dot{L_{sx}}$ because we are located on an arm, therefore during the transition L_{sx} is increasing for a minimal value of $\Delta_t * \dot{L_{sx}}^{min}$.

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