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Ahmed Doghmi
National Institute of Statistics and Applied Economics of Rabat, Morocco

Abderrahmane Ziad
Normandie University, Caen,
CREM CNRS UMR6211, France

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Ahmed Doghmi[ab*] and Abderrahmane Ziad [b]

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[a] *National Institute of Statistics and Applied Economics, Madinat Al Irfane, Rabat Institutes, 10100 Rabat, Morocco.*

[b] *University of Caen, Center for Research in Economics and Management (UMR CNRS 6211), 19 Rue Claude Bloch 14032 Caen, Cedex, France.*

Abstract. In this paper we explore the problem of Nash implementation providing new sufficient conditions called *I-monotonicity* and *I-weak no-veto power*. Firstly, we show that these conditions together with unanimity are sufficient for the implementation of social choice correspondences (SCCs) in Nash equilibria. Secondly, we prove that, in the domain of the private good economies with *single-plateaued* preferences, a solution of the problem of fair division is Nash implementable if and only if it satisfies Maskin monotonicity. We provide examples of SCCs satisfying or not Maskin monotonicity.

Keywords: Nash implementation; Private good economies; Single-plateaued preferences.
JEL classification: C72; D71

1 Introduction

The goal of the implementation theory is to study the problem of the asymmetric information between a planner (or social designer) and a set of individuals in a society. The planner hopes to maximize an objective function, called *social choice correspondence* (or *rule*), that represents the welfare of a society and provides the desired outcomes but the social designer confronts individuals who state the false preferences on the outcomes. So that the agents reveal their true preferences, the social designer organizes a non-cooperative game among agents. If the payment with the solution of this game corresponds to the socially desired alternative and vice versa, we say that the social choice correspondence (SCC) which give this socially desired alternative is implementable in the solution of the game. To achieve this objective, some conditions should be imposed on SCCs. Thus, Maskin (1977, 1999) was the first to give the necessary and almost sufficient conditions for the implementation of social choice correspondences in Nash equilibria. For

*Corresponding author. Tel.:+212 6 63 81 33 42. *E-mail addresses:* doghmi@insea.ac.ma, ahmeddoghmi@hotmail.com (A.Doghmi), abderrahmane.ziad@unicaen.fr (A. Ziad)

necessity, he showed that any Nash implementable SCC must satisfy a condition known now as *Maskin monotonicity*. This condition stipulates that if an alternative a is socially chosen in a profile R and if the alternatives ranked below a for all agents remain ranked below it (in large sense) in a new profile R', then the alternative a must be socially chosen in R'. Maskin proved that this condition alone is not sufficient. Thus, for sufficiency, he gave an additional condition called, *no-veto power*. This condition requires that if an alternative is ranked at the top for all agents except one, then this alternative must be socially chosen. Maskin showed that if there is at least three players participating at a mechanism, then any SCC satisfying Maskin monotonicity and no-veto power can be implemented in Nash equilibria.

However, the no-veto power condition is not checked in many economic and political applications. Sjöström (1991) and Danilov (1992) provided a full characterization for Maskin's result. Yamato (1992) generalized Danilov's result proposing an elegant condition for sufficiency called *strong monotonicity*[1].

Thomson (1990, 2010) applied these theoretical results in private good economies with single-peaked preferences. He showed that, only Pareto correspondence satisfies no-veto power, and so it is the unique correspondence which can be implemented by Maskin's Theorem while many other correspondences are implemented by appealing to strong monotonicity and Sjöström's algorithm (1991). Doghmi and Ziad (2008a) reexamined Maskin's result by providing new sufficient conditions. By applying these conditions to the domain of the private good economies with single-peaked preferences, Doghmi and Ziad (2008b) proved that all unanimous correspondences satisfying Maskin monotonicity can be implemented in Nash equilibria. Thus, by this very easy conditions compared to the different techniques used by Thomson (1990, 2010), they solved definitively the problem of implementation in the domain of the private good economies with single-peaked preferences. Now, the question is: What happens about Nash implementability in private good economies when preferences are *single-plateaued*?

To resolve this problem, we propose new sufficient conditions called *I-monotonicity* and *I-weak no-veto power* and we prove that this conditions together with unanimity are sufficient for Nash implementation. By applying this result to private good economies with single-plateaued preferences, we show that I-monotonicity alone is sufficient and becomes equivalent to Maskin monotonicity. We find that Maskin monotonicity alone is necessary and sufficient to implement the solutions of the problem of fair division in this domain. Surprisingly, we find that Pareto correspondence and its intersections with the no envy correspondence and with the individually rational correspondence from equal division do not satisfy Maskin monotonicity and so they are not Nash implementable.

The rest of this paper is organized as follows. Section 2 reviews the related literature to domains of single-plateaued preferences. In Section 3, we introduce notations and definitions. In Section 4, we provide new sufficient conditions. In Section 5, we give applications in the domain of private good economies with single-plateaued preferences. Section 6 provides our concluding remarks.

[1] Maskin's results are also extended by Moore and Repullo (1990), Dutta and Sen (1991) and several other authors that are not cited here.

2 Related literature to domains of single-plateaued preferences

By applying families of preference profiles that satisfy some special conditions, many positive results have been obtained in economic and political sciences. Perhaps the very known among these conditions is the notion of single peakedness mentioned in the above section. This notion is introduced by Black (1948) in social choice theory. It requires that each agent has a unique best alternative. However, although single-peakedness is a domain restriction that allows to have very nice and interesting results, this restriction on individual preferences is very strong. Because the fact that to have only one maximal element, without admitting the indifference with at least one other alternative, is not always natural. Thus, many authors have explored the consequences of admitting more than one maximal element over individual preferences. They have expanded the domain of single-peakedness into a domain well-known now as *single-plateauedness*, which allows agents to be indifferent among several best elements. Assume that there is an amount $\Omega \in \mathbb{R}_{++}$ of certain infinitely divisible good that is to be allocated among a set of n agents. The preference of each agent i is represented by a continuous and single-plateaued preference relation R_i over $[0, \Omega]$ as illustrated in figure 1.

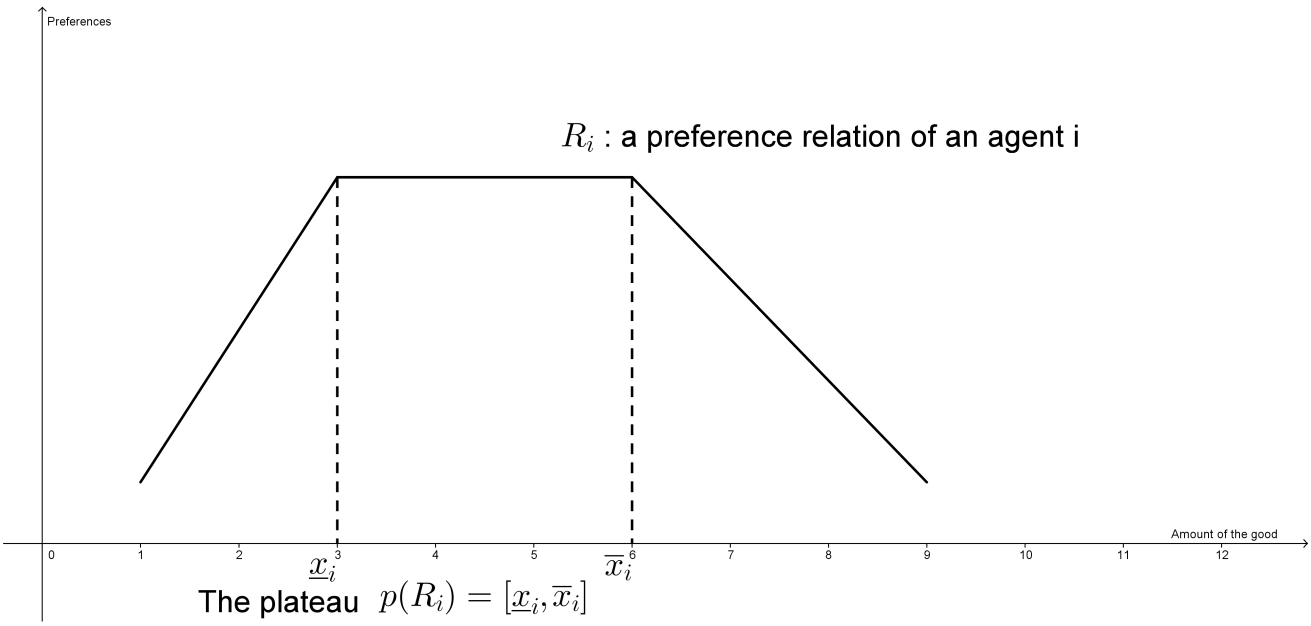

Figure 1: Single-plateaued preferences.

This domain has been explored by several authors in social choice theory and games theory. The more recent work is that of Bossert and Peters (2013). They examined the notion of single-plateauedness in a choice-theoretic setting. They showed that the single-plateaued choice is characterized by independence of irrelevant alternatives and a continuity properties. We can cite other many applications; for example, Moulin (1984) enlarged the domain of single-peakedness to that of single-plateauedness in order to characterize a class of generalized Condorcet-winners choice functions. Berga (1998, 2006) explored the problem of the provision of one pure public good. He characterized the class of strategy-proof voting schemes on single-plateaued preferences in generalizing Moulin's characterization (1980) of strategy-proof voting schemes on single-peaked preferences. Ehlers (2002) considered private good economies with single-plateaued preferences where a collective endowment of a perfectly divisible commodity has to be allocated among a finite set of agents. He characterized the class of sequential-allotment solutions in enlarging the family of solutions studied by Barberà , Jackson and Neme (1997). Barberà (2007) introduced the notion of single-plateaued preference profiles in extending Black's result (1948) and Moulin's result (1980,1984). Berga and Moreno (2009) are interested in the problem of the provision of one pure public good when agents have either single-plateaued preferences or single-peaked preferences over the set of alternatives. They studied the relationships between different non-manipulability notions under these two domains including the most important concept in the implementation theory: Maskin monotonicity.

This paper attempts to study the problem of implementation theory in the domain of single-plateaued preferences. In this area, a few works have addressed this issue. To our knowledge, only in a recent work, but in different purposes, Lombardi and Yoshihara (2011) have examined the implementability of the Pareto correspondence in a model of the private good economies when preferences are single-plateaued. Using Moore and Repullo's conditions (1990), they showed that this correspondence becomes not implementable when the domain of individual preferences is enlarged from single-peakedness to single-plateaued preference profiles. In our work, we use simple new conditions and we study the implementability of a family of the solutions of the problem of fair division.

3 Notations and definitions

Let A be a set of alternatives, and let $N = \{1, ..., n\}$ be a set of individuals, with generic element i. Each individual i is characterized by a preference relation R_i defined over A, which is a complete, transitive, and reflexive relation in some class $\Re_i$ of admissible preference relations. Let $\Re = \Re_1 \times ... \times \Re_n$. An element $R = (R_1, ..., R_n) \in \Re$ is a preference profile. The relation R_i indicates the individual's i preference. For $a, b \in A$, the notation aR_ib means that the individual i weakly prefers a to b. The asymmetrical and symmetrical parts of R_i are noted respectively by P_i and $\sim_i$.
A social choice correspondence (SCC) F is a multi-valued mapping from $\Re$ into $2^A \setminus \{\emptyset\}$, that associates with every R a nonempty subset of A. For all $R_i \in \Re_i$ and all $a \in A$, the lower contour set for agent i at alternative a is noted by: $L(a, R_i) = \{b \in A \mid aR_ib\}$. The strict lower contour set and the indifference lower contour set are noted respectively by $LS(a, R_i) = \{b \in A \mid aP_ib\}$ and $LI(a, R_i) = \{b \in A \mid a \sim_i b\}$.

A mechanism (or form game) is given by $\Gamma = (S, g)$ where $S = \Pi_{i\in N}S_i$; S_i denotes the strategy set of the agent i and g is a function from S to A. The elements of S are denoted by $s = (s_1, s_2, ..., s_n) = (s_i, s_{-i})$, where $s_{-i} = (s_1, ..., s_{i-1}, s_{i+1}, ..., s_n)$. When $s \in S$ and $b_i \in S_i$, $(b_i, s_{-i}) = (s_1, ..., s_{i-i}, b_i, s_{i+1}, ..., s_n)$ is obtained after replacing s_i by b_i, and $g(S_i, s_{-i})$ is the set of results which agent i can obtain when the other agents choose s_{-i} from $S_{-i} = \Pi_{j\in N, j\neq i}S_j$.
A Nash equilibrium of the game (Γ, R) is a vector of strategies $s \in S$ such that for any i, $g(s)R_ig(b_i, s_{-i})$ for all $b_i \in S_i$, i.e. when the other players choose s_{-i}, the player i

cannot deviate from s_i. Given $N(g,R,S)$ the set of Nash equilibria of the game (Γ,R), a mechanism $\Gamma=(S,g)$ implements an SCC F in Nash equilibria if for all $R\in\Re$, $F(R)=g(N(g,R,S))$.
We say that an SCC F is implementable in Nash equilibria if there is a mechanism which implements it in these equilibria.
A SCC F satisfies unanimity if for any $a\in A$ and any $R\in\Re$, if for any $i\in N$, $L(a,R_i)=A$, then $a\in F(R)$.

Maskin (1977/1999) introduced the following conditions on F to characterize the SCC's that are implementable in Nash equilibria.
Monotonicity: A SCC F satisfies monotonicity if for all $R,R'\in\Re$, for any $a\in F(R)$, if for any $i\in N$, $L(a,R_i)\subseteq L(a,R'_i)$, then $a\in F(R')$.
No-veto power: A SCC F satisfies no-veto power if for i, $R\in\Re$, and $a\in A$, if $L(a,R_j)=A$ for all $j\in N\backslash\{i\}$, then $a\in F(R)$.
Maskin (1977/1999) proved that any Nash implementable correspondence satisfies Maskin monotonicity and for at leat three players any social choice correspondence satisfies Maskin monotonicity and no-veto power is Nash implementable.

Yamato (1992) generalized the Danilov's result (1992) by proposing a strong version of monotonicity for weak preferences. To define this property, let us introduce the following notion. Let i be a player and $B\subset A$. An alternative $b\in B$ is essential for i in set B if $b\in F(R)$ for some preference profile R such that $L(b,R_i)\subset B$. The set of all essential elements is denoted as $Ess_i(F,B)$.
Strong monotonicity: A SCC F satisfies strong monotonicity if for all $R,R'\in\Re$ and for all $a\in F(R)$, if for all $i\in N$, $Ess_i(F,L(a,R_i))\subset L(a,R'_i)$, then $a\in F(R')$.
Yamato (1992) proved that any SCC satisfying strong monotonicity is Nash implementable with at least three agents. This condition becomes necessary under certain mild condition imposed on admissible preferences.

Doghmi and Ziad (2008a) reexamined Maskin's result (1977/1999) by introducing the following new sufficient conditions.
Strict monotonicity: A SCC F satisfies strict monotonicity if for all $R,R'\in\Re$, for any $a\in F(R)$, if for any $i\in N$, $LS(a,R_i)\cup\{a\}\subseteq L(a,R'_i)$, then $a\in F(R')$.
Strict weak no-veto power: A SCC F satisfies strict weak no-veto power if for i, $R\in\Re$, and $a\in F(R)$, if for $R'\in\Re$, $b\in LS(a,R_i)\subseteq L(b,R'_i)$ and $L(b,R'_j)=A$ for all $j\in N\backslash\{i\}$, then $b\in F(R')$.
Doghmi and Ziad (2008a) showed that, in addition to unanimity, the conditions of strict monotonicity and strict weak no-veto power are sufficient for an SCC to be implementable.

4 Sufficient Conditions

Now, we present new sufficient conditions called *I-monotonicity* and *I-weak no-veto power*. We show that these conditions together with unanimity are sufficient for the implementation of social choice correspondences in Nash equilibria as long as there are at least three players.

We begin by defining a subset of indifferent options.

Definition 1 *(Indifferent options subset)*
For any agent's i preference R_i, any alternative $a \in F(R)$, for some singleton "operator" $\{o\} \in LI(a, R_i)$ with $o \neq a$, the indifferent options subset is the subset $I(a, o, R_i) = \{b \in A \setminus \{a, o\}$ s.t. $a \sim_i b \sim_i o\}$.

This subset denotes, according to the operator o, the set of all elements in the indifference class of a (not including a and o) under R_i provided that the indifference class of a contains at least three alternatives. To illustrate this subset, suppose that we have the set of alternatives $A = \{a, b, c, d, e\}$ with $a \in F(R)$, and an agent i's preference

$R_i \in \Re_i$ on A such that,

R_i
e
a,b,c,d
f

We have $LI(a, R_i) = \{a, b, c, d\}$ and so the operator $\{o\}$ can be b, c or d. Therefore we have either $I(a, o = b, R_i) = \{c, d\}$ or $I(a, o = c, R_i) = \{b, d\}$ or $I(a, o = d, R_i) = \{b, c\}$.

For this subset of indifferent options, we give the following remark.

Remark 1 *$I(a, o, R_i) \neq \emptyset$ if $|LI(a, R_i)| \geq 3$, otherwise $I(a, o, R_i) = \emptyset$.*

To define our first sufficient condition called *I-monotonicity*, we do not need to consider all elements of the indifference class $LI(a, R_i) \setminus \{a\}$, but there should simply be one element of this class ($\exists o \in LI(a, R_i) \setminus \{a\}$) for which the inclusion of the condition is checked. This element can be different from a player to another.

Definition 2 *(I-monotonicity)*
A SCC F satisfies I-monotonicity if for all $R, R' \in \Re$, for any $a \in F(R)$, if for any $i \in N$, $LS(a, R_i) \cup I(a, o, R_i) \cup \{a\} \subseteq L(a, R'_i)$ for some $o \in LI(a, R_i) \setminus \{a\}$, then $a \in F(R')$.

Roughly speaking, I-monotonicity means that if an alternative a is socially chosen in a profile R and if, for all agents, the alternatives set ranked strictly below a (including a) in addition to the *indifferent options subset* remain ranked below a (in large sense) in a new profile R', then the alternative a must be socially chosen in R'. Generally, I-monotonicity implies Maskin monotonicity.

Example 1: $A = \{a, b, c, d, e, f\}$, $N = \{1, 2, 3\}$ and $\Re = \{R, R'\}$ are defined by:

R:

R_1	R_2	R_3
d,e	b	b,d
a,c	c	a,c
f	a,d,f	f
b	e	e

R':

R'_1	R'_2	R'_3
a,b,c,d,e	b,d	a,b
f	c	c,d
	a,e,f	e,f

$F(R) = \{a, f\}$ $\qquad$ $F(R') = \{a, b, c\}$

We have $f \in F(R)$ and for player 1 we have $I(f, o = \emptyset, R_1) = \{\emptyset\}$, and hence $LS(f, R_1) \cup I(f, R_1) \cup \{f\} = \{b, f\} \nsubseteq L(f, R'_1) = \{f\}$.

In the inverse sense, we have $b \in F(R')$ and $b \notin F(R)$. For players 2 and 3, we have $LS(b, R'_{i=2,3}) \cup I(b, o = \emptyset, R'_{i=2,3}) \cup \{b\} \subseteq L(b, R_{i=2,3})$. For player 1, we have $LI(b, R'_1) \setminus \{b\} = \{a, c, d, e\}$, hence we should have *the non-inclusion* (i.e., $LS(b, R'_1) \cup I(b, o, R'_1) \nsubseteq L(b, R_1)$) for all $o \in LI(b, R'_1) \setminus \{b\}$. We have either $I(b, o = a, R'_1) = \{c, d, e\}$ or $I(b, o = c, R'_1) = \{a, d, e\}$ or $I(b, o = d, R'_1) = \{a, c, e\}$ or $I(b, o = e, R'_1) = \{a, d, c\}$. If we have $I(b, o = a, R'_1) = \{c, d, e\}$, $LS(b, R'_1) \cup I(b, a, R'_1) \cup \{b\} = \{b, c, d, e, f\} \nsubseteq L(b, R_1) = \{b\}$. If we have $I(b, o = c, R'_1) = \{a, d, e\}$, $LS(b, R'_1) \cup I(b, c, R'_1) \cup \{b\} = \{a, b, d, e, f\} \nsubseteq L(b, R_1) = \{b\}$. If we have $I(b, o = d, R'_1) = \{a, c, e\}$, $LS(b, R'_1) \cup I(b, d, R'_1) \cup \{b\} = \{a, b, c, e, f\} \nsubseteq L(b, R_1) = \{b\}$. If we have $I(b, o = e, R'_1) = \{a, d, c\}$, $LS(b, R'_1) \cup I(b, e, R'_1) \cup \{b\} = \{a, b, c, d, f\} \nsubseteq L(b, R_1) = \{b\}$.

Finally, we have $c \in F(R')$ and $c \notin F(R)$. For players 2 and 3, we have $LS(c, R'_{i=2,3}) \cup I(c, o = \emptyset, R'_{i=2,3}) \cup \{c\} \subseteq L(c, R_{i=2,3})$. For player 1, we have $LI(c, R'_1) \setminus \{c\} = \{a, b, d, e\}$, hence we should have the non-inclusion (i.e., $LS(c, R'_1) \cup I(c, o, R'_1) \nsubseteq L(c, R_1)$) for all $o \in LI(c, R'_1) \setminus \{c\}$. We have either $I(c, o = a, R'_1) = \{b, d, e\}$ or $I(c, o = b, R'_1) = \{a, d, e\}$ or $I(c, o = d, R'_1) = \{a, b, e\}$ or $I(c, o = e, R'_1) = \{a, b, d\}$. If we have $I(c, o = a, R'_1) = \{b, d, e\}$, $LS(c, R'_1) \cup I(c, a, R'_1) \cup \{c\} = \{b, c, d, e, f\} \nsubseteq L(c, R_1) = \{a, b, c, f\}$. If we have $I(c, o = b, R'_1) = \{a, d, e\}$, $LS(c, R'_1) \cup I(c, b, R'_1) \cup \{c\} = \{a, c, d, e, f\} \nsubseteq L(c, R_1) = \{a, b, c, f\}$. If we have $I(c, o = d, R'_1) = \{a, b, e\}$, $LS(c, R'_1) \cup I(c, d, R'_1) \cup \{c\} = \{a, b, c, e, f\} \nsubseteq L(c, R_1) = \{a, b, c, f\}$. If we have $I(c, o = e, R'_1) = \{a, b, d\}$, $LS(c, R'_1) \cup I(c, e, R'_1) = \{a, b, c, d, f\} \nsubseteq L(c, R_1) = \{a, b, c, f\}$. Thus, F satisfies I-monotonicity.

Now, we introduce our second condition, called *I-weak no-veto power.*

Definition 3 *(I-weak no-veto power)*
A SCC F satisfies I-weak no-veto power if for i, $R \in \Re$, and $a \in F(R)$, if for $R' \in \Re$, $b \in LS(a, R_i) \cup I(a, o, R_i) \subseteq L(b, R'_i)$ and $L(b, R'_j) = A$ for all $j \in N \backslash \{i\}$, for some $o \in LI(a, R_i) \setminus \{a\}$, then $b \in F(R')$.

In words, I-weak no-veto power means that if an alternative a is socially chosen in a profile R and for an agent i, if an alternative b, belongs to the subset of the alternatives which are ranked strictly below a or it belongs to the *indifferent options subset*, becomes weakly preferred to these alternatives in a new profile R' for i, and it will be ranked at the top for all $j \neq i$, then the alternative b must be socially chosen in R'.

In example 1, the I-weak no-veto power condition is checked. We have in profile R, $a \in F(R)$ and for player 1, $LI(a, R_1) = \{a, c, \}$; therefore $\mid LI(a, R_1) < 3 \mid$ and so $I(a, o = d, R_1) = \emptyset$. We have $b \in LS(a, R_1) \cup I(a, o, R_1) = \{b, f\} \subseteq L(b, R'_1) = A$. For players 2 and 3, $L(b, R'_{i=2,3}) = A$ and so $b \in F(R')$ as required. In the inverse sense, we have $L(b, R_{i=2,3}) = A$, for $a \in F(R')$, we have for player 1, $LI(a, R'_1) \setminus \{a\} = \{b, c, d, e\}$ and $b \in LS(a, R'_1) \cup I(a, o, R'_1) \nsubseteq L(b, R_1)$ for all $o \in LI(a, R'_1) \setminus \{a\}$. We follow the same reasoning for the other elements of $F(R')$.

Remark 2 *If $I(a, o, R_i) = \emptyset$, then I-monotonicity and I-weak no-veto power become, respectively, equivalent to strict monotonicity and strict weak no-veto power.*

In private good economies with single-peaked and single-dipped preferences, $|LI(a, R_i)| < 3$. By Remarks 1 and 2, the condition of I-monotonicity is equivalent to strict monotonicity and the condition of I-weak no-veto power and strict weak no-veto power are equivalent in these domains.

In the following theorem, we give our first main result in this paper.

Theorem 1 *Let $n \geq 3$. If an SCC F satisfies I-monotonicity, I-weak no-veto power and unanimity, then F can be implemented in Nash equilibria.*

Proof. See appendix.

In Example 1, for Maskin's conditions (1977,1999), while F is monotonic, this SCC does not satisfy no-veto power. In profile R, alternative b is not chosen by F even if this alternative is top-ranked by players 2 and 3. For Yamato's condition (1992), strong monotonicity is not checked. We have $f \in F(R), Ess_1(F, L(f, R_1)) = \{f\} \subseteq L(f, R'_1) = \{f\}$, $Ess_2(F, L(f, R_2)) = \{a, f\} \subseteq L(f, R'_2) = \{a, e, f\}$, and $Ess_3(F, L(f, R_3)) = \{f\} \subseteq L(f, R'_3) = \{e, f\}$, but $f \notin F(R')$. For Doghmi and Ziad's conditions, F does not satisfy strict monotonicity. We have $c \in F(R'), LS(c, R'_1) \cup \{c\} = \{c, f\} \subseteq L(c, R_1) = \{a, b, c, f\}$, $LS(c, R'_2) \cup \{c\} = \{a, c, e, f\} \subseteq L(c, R_2) = \{a, c, d, e, f\}$ and $LS(c, R'_3) \cup \{c\} = \{c, e, f\} \subseteq L(c, R_3) = \{a, c, e, f\}$, but $c \notin F(R)$. Thus, Maskin's Theorem, Yamato's result (1992) or Doghmi and Ziad's result (2008 a) are silent about the implementability of F. We see that the unanimity condition is checked by F and we have showed that F satisfies I-monotonicity and I-weak no-veto power, and hence is Nash implementable by Theorem 1.

5 Applications to private good economies with single-plateaued preferences

There is an amount $\Omega \in \mathbb{R}_{++}$ of certain infinitely divisible good that is to be allocated among a set $N = \{1, ..., n\}$ of n agents. The preference of each agent $i \in N$ is represented by a continuous[2] and single-plateaued preference relation R_i over $[0, \Omega]$ (the asymmetrical part is written P_i and the symmetrical part $\sim_i$). For all $x_i, y_i \in [0, \Omega]$, $x_i R_i y_i$ means that, for the agent i, to consume a share x_i is as good as to consume the quantity y_i. A feasible allocation for the economy (R, Ω) is a vector $x \equiv (x_i)_{i \in N} \in \mathbb{R}^n_+$ such that $\sum_{i \in N} x_i = \Omega$ and X is the set of the feasible allocations. We note that the feasible allocations set is $X = [0, \Omega] \times ... \times [0, \Omega]$. Thus, $L(x, R_i) = X$ is equivalent to $L(x_i, R_i) = [0, \Omega]$. For the set $L(x, R_i) = X$, xR_iy for all $y \in X$ implies that $x_iR_iy_i$. Thus, the agents preferences are defined over individual consumption spaces, not over allocation space. Then the proprieties of implementation theory, presented in general setup in Section 2, become as follows. A SCC F satisfies monotonicity if for all $R, R' \in \Re$, for any $x \in F(R)$, if for any $i \in N$, $L(x_i, R_i) \subseteq L(x_i, R'_i)$, then $x \in F(R')$. A SCC F satisfies I-monotonicity if for all $R, R' \in \Re$, and for any $x \in F(R)$, if for any $i \in N$, $LS(x_i, R_i) \cup I(x_i, z_i, R_i) \cup \{x_i\} \subseteq L(x_i, R'_i)$ for some $z \in LI(x, R_i) \setminus \{x\}$, then $x \in F(R')$. A SCC F satisfies no-veto power

[2]Continuous here means that if $[a, b[\cup]b, c] \subseteq L(x_i, R'_i)$ for some a, b, c, x and R', then $[a, c] \subseteq L(x_i, R'_i)$.

if for i, $R \in \Re$, and $x \in X$, if $L(x_j, R_j) = [0, \Omega]$ for all $j \in N \backslash \{i\}$, then $x \in F(R)$. A SCC F satisfies I-weak no-veto power if for i, $R \in \Re$, $x, y, z \in X$, and $x \in F(R)$, if for $R' \in \Re$, $y_i \in LS(x_i, R_i) \cup I(x_i, z_i, R_i) \subseteq L(y_i, R'_i)$ and $L(y_j, R'_j) = [0, \Omega]$ for all $j \in N \backslash \{i\}$, for some $z \in LI(x, R_i) \setminus \{x\}$, then $y \in F(R')$. A SCC F satisfies unanimity if for any $x \in X$ and any $R \in \Re$, if for any $i \in N$, $L(x_i, R_i) = [0, \Omega]$, then $x \in F(R)$.

We note that the free disposability of the good is not assumed.

A preference relation R_i is *single-plateaued* if there are two numbers $\underline{x}_i, \overline{x}_i \in [0, \Omega]$ such that $\underline{x}_i \leq \overline{x}_i$ and for all $x_i, y_i \in [0, \Omega]$: (i) if $y_i < x_i \leq \underline{x}_i$ or $\overline{x}_i \leq x_i < y_i$, then $x_i P_i y_i$; (ii) We call $p(R_i) \equiv [\underline{x}_i, \overline{x}_i]$ the *plateau* of R_i, $\underline{x}$ is the left end-point of the plateau of R_i, and $\overline{x}$ is the right end-point. A preference relation R_i is *single-peaked* if $\underline{x}_i = \overline{x}_i$.

The class of all single-plateaued preference relations is represented by $\Re_{sp} \subseteq \Re$. For $R \in \Re_{sp}$, let $p(R) = (p(R_1), ..., p(R_n))$ be the profile of plateaus (or of preferred consumptions). A single plateaued preference relation $R_i \in \Re_{sp_i}$ is described by the function $r_i : [0, \Omega] \to [0, \Omega]$ which is defined as follows: $r_i(x_i)$ is the consumption of the agent i on the other side of the plateau which is indifferent to x_i (if it exists), else, it is 0 or Ω. Formally, if $x_i \leq p(R_i)$, then, $r_i(x_i) \geq p(R_i)$ and $x_i \sim_i r_i(x_i)$ if such a number exists or $r_i(x_i) = \Omega$ otherwise; if $x_i \geq p(R_i)$, then, $r_i(x_i) \leq p(R_i)$ and $x_i \sim_i r_i(x_i)$ if such a number exists or $r_i(x_i) = 0$ otherwise.

Let us introduce some known correspondences.

No-Envy correspondence, NE, (Foley, 1967). This correspondence selects the feasible allocations for which each agent prefers his own share than the shares of the other agents. It is defined as follows: Let $R \in \Re_{sp}$, $NE(R) = \{x \in X \text{ if } x_i R_i x_j \text{ for all } i, j \in N\}$.

Individually Rational Correspondence from Equal Division, I_{ed}: This correspondence selects the feasible allocations for which each agent prefers his own share to the average one. It is defined as follows: Let $R \in \Re_{sp}$, $I_{ed}(R) = \{x \in X : x_i R_i (\Omega/n)$ for all $i \in N\}$.

Pareto correspondence, P: This solution selects the feasible allocations which are not weakly dominated by an other allocation for all agents and not strictly dominated for at least one agent. It is defined as follows: Let $R \in \Re_{sp}$, $P(R) = \{x \in X : \nexists x' \in X$ such that for all $i \in N$, $x'_i R_i x_i$, and for some $i \in N$, $x'_i P_i x_i\}$.

5.1 Some sufficient conditions and robustness of Maskin monotonicity

In this subsection, we provide our second main result of the paper. We prove that an SCC in the domain of the private good economies is Nash implementable if and only if it satisfies Maskin monotonicity. For this, we show that I-monotonicity alone is sufficient for Nash implementability and becomes equivalent to Maskin monotonicity. To prove this, we give the following Lemmas.

Lemma 1 *Let $R, R' \in \Re_{sp}$ and $x, y, z \in X$. If the preferences are single-plateaued, $y_i \in LS(x_i, R_i) \cup I(x_i, z_i, R_i)$, and $LS(x_i, R_i) \cup I(x_i, z_i, R_i) \subseteq L(y_i, R'_i)$, then $L(y_i, R'_i) = [0, \Omega]$.*

The proof of Lemma 1 is omitted, it follows the same reasoning in Lemma 1 of Doghmi and Ziad (2008 b).

Assumption 1 *For some SCC F, for all $x \in X$, there is a profile $R \in \Re$ such that $x \in F(R)$.*[3]

Lemma 2 *On a single-plateaued domain, any I-monotonic SCC satisfies unanimity.*

Proof. Suppose not. Let $x \in X$ and any $\widetilde{R} \in \Re_{sp}$, for any $i \in N$, $[0, \Omega] = L(x_i, \widetilde{R}_i)$, and $x \notin F(\widetilde{R})$. By Assumption 1, for all $x \in X$, there is a profile $R \in \Re$ such that $x \in F(R)$ and so for all $i \in N$, $LS(x_i, R_i) \cup I(x_i, z_i, R_i) \cup \{x_i\} \subseteq [0, \Omega] = L(x_i, \widetilde{R}_i)$. By I- monotonicity, $x \in F(\widetilde{R})$, a contradiction. Q.E.D.

According to Lemmas 1 and 2, we have the following corollary:

Corollary 1 *On a single-plateaued domain, any I-monotonic SCC satisfies I-weak no-veto power.*

Now, we show that I-monotonicity, alone, is sufficient for Nash implementation when preferences are single-plateaued.

Proposition 1 *Let $n \geq 3$. In the private good economies with single-plateaued preferences, any SCC satisfying I-monotonicity can be implemented in Nash equilibria.*

Proof. By Lemma 2, Corollary 1 and Theorem 1, the proof is completed as required. Q.E.D.

We prove that I-monotonicity is equivalent to Maskin monotonicity in the area of private good economies with single-plateaued preferences.

Proposition 2 *In the private good economies with single-plateaued preferences, I-monotonicity becomes equivalent to Maskin monotonicity.*

Proof. Let $R, R' \in \Re_{sp}$ and $x, y \in X$. Suppose that $x_i \leq \overline{x}_i$ (similar statements can be proved for $x_i > \overline{x}_i$). $i)\Rightarrow$, it is clair that $LS(x_i, R_i) \cup I(x_i, z_i, R_i) \cup \{x_i\} \subseteq L(x_i, R_i)$. Therefore, I- monotonicity implies Maskin monotonicity. $ii)$ $\Leftarrow$, in this case, suppose that $LS(x_i, R_i) \cup I(x_i, z_i, R_i) \cup \{x_i\} \subseteq L(x_i, R'_i)$ (1). We have two cases to study.

$i)$ If $x_i \notin [\underline{x}_i, \overline{x}_i]$ then $I(x_i, z_i, R_i) = \emptyset$ and so I-monotonicity becomes strict monotonicity.

$ii)$ If $x_i \in [\underline{x}_i, \overline{x}_i]$ and $\exists\{z_i\} \in p(R_i)$ s.t. $\underline{x}_i < z_i < \overline{x}_i$, then $\emptyset \neq I(x_i, z_i, R_i) = [\underline{x}_i, x_i[\cup]x_i, z_i[\cup]z_i, \overline{x}_i]$ and $LS(x_i, R_i) = [0, \underline{x}_i[\cup]\overline{x}_i, \Omega]$. Thus, by (1), $LS(x_i, R_i) \cup I(x_i, z_i, R_i) \cup \{x_i\} = [0, z_i[\cup]z_i, \Omega] \subseteq L(x_i, R'_i)$. By the continuity of preferences, $[0, \Omega] \subseteq L(x, R'_i)$ and so $L(x_i, R_i) \subseteq [0, \Omega] \subseteq L(x, R'_i)$. Thus, we have the inclusion of Maskin monotonicity. Q.E.D.

Through propositions 1 and 2, we complete the proof of the second main Theorem of the paper.

Theorem 2 *: Let $n \geq 3$. A SCC in the private good economies with single-plateaued preferences is Nash implementable if and only if satisfies Maskin monotonicity.*

To support this result, we give in the following subsection a series of SCCs' examples that satisfy or not Maskin monotonicity.

[3]This assumption is not checked when the SCCs are constants like the equal division correspondence.

5.2 Examples of SCC's satisfying or not Maskin monotonicity

In this subsection, we check whether or not the correspondences cited above and their intersections can be implemented in private good economies with *single-plateaued* preferences by Theorem 2. We begin by examining the implementability of the no-envy correspondence.

Proposition 3 *In the private good economies with single-plateaued preferences, the No-Envy correspondence satisfies Maskin monotonicity.*

The proof of this proposition is omitted. This, because the no-envy correspondence satisfies Maskin monotonicity in general domain and so this correspondence obviously satisfies this condition in the restricted domain. The same for the next Proposition, we omit the proof; the individually rational correspondence from equal division satisfies Maskin monotonicity in restricted domain while this correspondence is monotonic in general domain.

Proposition 4 *In the private good economies with single-plateaued preferences, the Individually Rational Correspondence from Equal Division satisfies Maskin monotonicity.*

The intersection of the no-envy correspondence with the individually rational correspondence from equal division satisfies Maskin monotonicity as shown in the following proposition.

Proposition 5 *The* $(NE \cap I_{ed})$ *correspondence satisfies Maskin monotonicity.*

Since Maskin monotonicity is stable under intersection, the proof of this Proposition is immediate from Propositions 2 and 3.

The Pareto correspondence does not satisfy Maskin monotonicity in general domain. In the next, we examine the implementability of this correspondence in the domain of the private good economies with single-plateaued preferences. The following proposition shows that this correspondence does not satisfy Maskin monotonicity and hence not Nash Implementable.[4]

Proposition 6 *In the private good economies with single-plateaued preferences, the Pareto correspondence does not satisfy Maskin monotonicity*

Proof. Let $R, R' \in \mathscr{D}_{sp}$, $x = (4, 8, 0), y = (4.5, 7.5, 0) \in X$ and $\sum_{i=1}^{3} x_i = \sum_{i=1}^{3} y_i = \Omega = 12$. Let $R_1 = R_1'$, $R_3 = R_3'$, $p(R) = (5, 10, 0)$, and $p(R') = (5, [6, 9], 0)$. Figure 2 illustrates such representations. Note that $x \in P(R)$ and for all $i \in N$, $L(x_i, R_i) \subseteq L(x_i, R_i')$. However, for profile R', we have $y_{i=2,3} \sim'_{i=2,3} x_{i=2,3}$ and $y_1 P_1' x_1$. Therefore, $x \notin P(R')$. Q.E.D.

[4] We note that the Pareto correspondence satisfies Maskin monotonicity and no-veto power condition in private good economies with *single-peaked* preferences. Hence, in this domain it is Nash implementable by Maskin's original result. For more detail, see Thomson (1990,2010), and Doghmi and Ziad (2008 b).

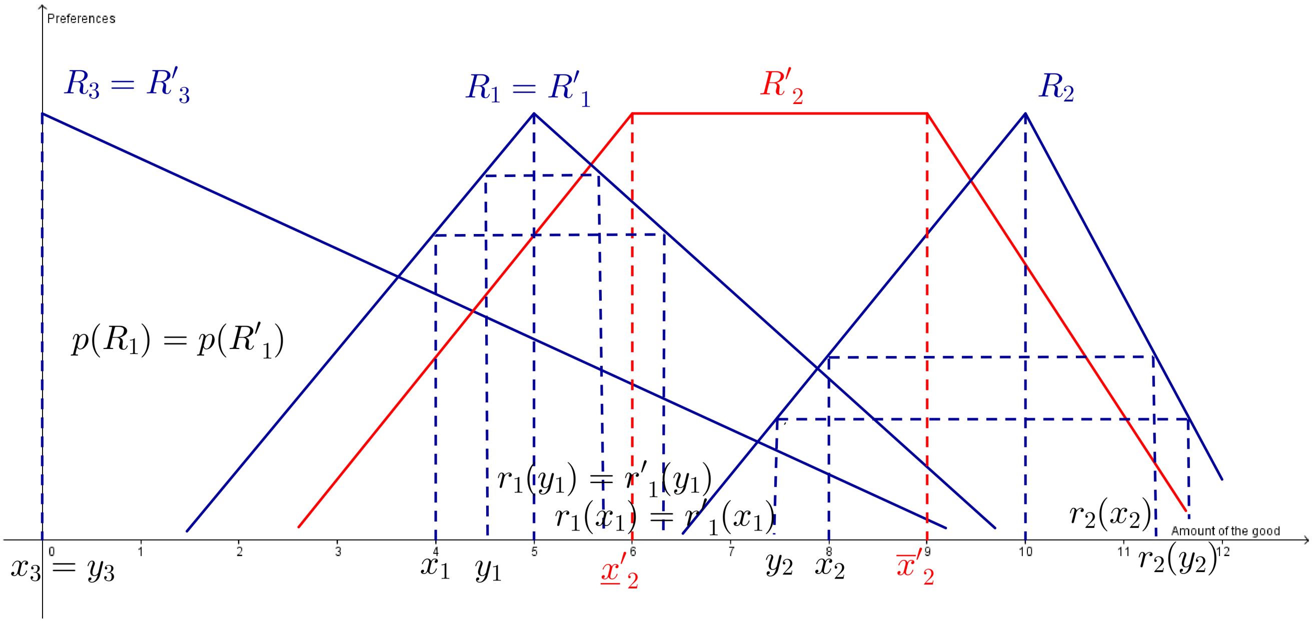

Figure 2: The Pareto correspondence and the $(NE \cap P)$ correspondence do not satisfy Maskin monotonicity.

In an independent work, Lombardi and Yoshihara (2011) also showed that Pareto correspondence does not satisfy Moore and Repullo's conditions and derive a similar result to our's.

The next proposition shows that the intersection of the no-envy solution with the Pareto solution does not satisfy Maskin monotonicity.

Proposition 7 *In the private good economies with single-plateaued preferences, the $(NE \cap P)$ correspondence does not satisfy Maskin monotonicity.*

The proof of this Proposition is omitted, it is similar to that of Proposition 6.

The intersection of the Individually Rational Correspondence from Equal Division with the Pareto correspondence does not satisfy Maskin monotonicity.

Proposition 8 *In the private good economies with single-plateaued preferences, the $(I_{ed} \cap P)$ correspondence does not satisfy Maskin monotonicity.*

Proof. Let $R, R' \in \mathscr{D}_{sp}$, $x = (1, 9, 2), y = (3, 6, 3) \in X$ and $\sum_{i=1}^{3} x_i = \sum_{i=1}^{3} y_i = \Omega = 12$. Let $R_1 = R_3$, $R'_1 = R'_3$, $p(R) = ([0, 2], [5.75, 7.45], [0, 2])$, and $p(R') = ([0, 3.25], [5.5, 8.5], [0, 3.25])$. Figure 3 illustrates such representations.

We have for all $i \in 1, 2, 3$, $x_i R_i \frac{\Omega}{n}$, and $x_1 P_1 y_1$, $y_2 P_2 x_2$ and $x_3 P_3 y_3$. Therefore, $x \in (P \cap I_{ed})(R)$. We have for all $i \in N$, $L(x_i, R_i) \subseteq L(x_i, R'_i)$. However, in profile R', we have $y_{i=1,3} \sim'_{i=1,3} x_{i=1,3}$ and $y_2 P'_2 x_2$. Therefore, $x \notin P(R')$ and so $x \notin (P \cap I_{ed})(R')$. Q.E.D.

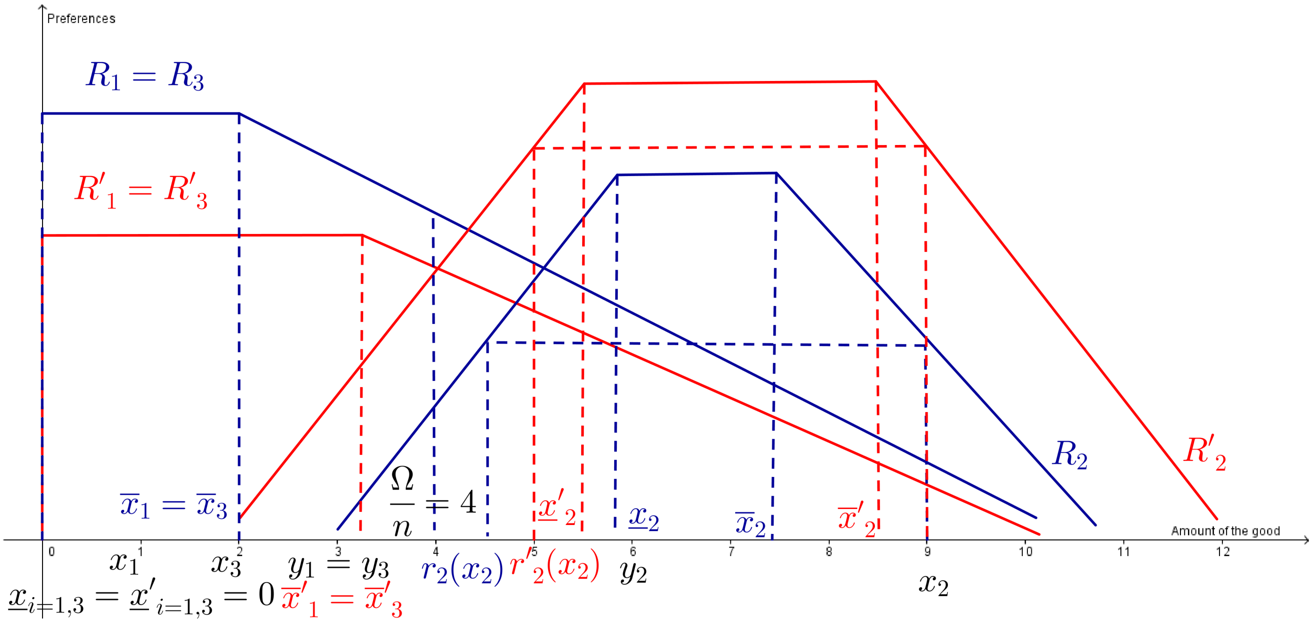

Figure 3: The $(I_{ed} \cap P)$ correspondence does not satisfy Maskin monotonicity.

Corollary 2 *In the domain of the private good economies with single-plateaued preferences, the Pareto correspondence and its intersections with the No- Envy correspondence and with the Individually Rational Correspondence from Equal Division can not be implemented in Nash equilibria.*

5.3 Discussion: why I-monotonicity and not strict monotonicity

In this subsection, we will focus our discussion on the strict monotonicity as long as this condition is an important tool for Nash implementation in the case of single-peaked preferences. In fact, Doghmi and Ziad (2008a) showed that strict monotonicity together with strict weak no veto and unanimity is sufficient for the implementation of social choice correspondences in Nash equilibria. By applying these conditions to the domain of the private good economies with single-peaked preferences, the authors proved that strict monotonicity alone is sufficient for Nash implementability and becomes equivalent to Maskin monotonicity. In the following, we prove that the enlargement of this result to the domain of the private good economies with single-plateaued preferences does not work. We show that strict monotonicity alone is sufficient for Nash implementability, but it is not equivalent to Maskin monotonicity. To prove this, we begin by the following results.

Lemma 3 *Let $R, R' \in \Re_{sp}$ and $x, y \in X$. If the preferences are single-plateaued, $y_i \in LS(x_i, R_i)$, and $LS(x_i, R_i) \subseteq L(y_i, R'_i)$, then $L(y_i, R'_i) = [0, \Omega]$.*

Proof. The proof is similar to that of Doghmi and Ziad (2008 b), it is omitted.

By the same reasoning in Subsection 5.1, we provide the following results:

Lemma 4 *On a single-plateaued domain, any strict monotonic SCC satisfies unanimity.*

According to Lemmas 3 and 4, we have the following corollary:

Corollary 3 *On a single-plateaued domain, any strict monotonic SCC satisfies strict weak no-veto power.*

By using Lemma 4, Corollary 3, and Theorem 1 of Doghmi and Ziad (2008 a), we complete the Proof of the following Proposition:

Proposition 9 *Let $n \geq 3$. In private good economies with single plateaued preferences, any SCC satisfying strict monotonicity can be implemented in Nash equilibria.*

Nevertheless, we show in the next proposition that, contrary to the domain of the private good economies with single-peaked preferences, when preferences are single-plateaued, strict monotonicity is not equivalent to Maskin monotonicity.

Proposition 10 *In private good economies with single-plateaued preferences, the strict monotonicity is not equivalent to Maskin monotonicity.*

Proof. Let $R, R' \in \Re_{sp}$ and $x, y \in X$. Suppose that $x_i \leq \overline{x}_i$ (similar statements can be proved for $x_i > \overline{x}_i$). $i)\Rightarrow$, it is clair that $LS(x_i, R_i) \cup \{x_i\} \subseteq L(x_i, R_i)$. Therefore, strict monotonicity implies Maskin monotonicity. $ii)$ $\Leftarrow$, in this case, suppose that $LS(x_i, R_i) \cup \{x_i\} \subseteq L(x_i, R'_i)$ (1). If $x_i \in p(R_i)$ and $x_i \notin p(R'_i)$, then, from (1) it is clear that $L(x_i, R_i) = [0, \Omega] \nsubseteq L(x_i, R'_i)$. Thus, the inclusion of Maskin monotonicity is not checked. Q.E.D.

Since strict monotonicity is not equivalent to Maskin monotonicity, we will return to Proposition 9 to examine whether this sufficient condition is checked or not by the monotonic SCC under consideration in the domain of the private good with single-plateaued preferences. We give the following propositions.

Proposition 11 *The Individually Rational Correspondence from Equal Division does not satisfy strict monotonicity.*

Proof. Let $R, R' \in \Re_{sp}$ and $N = \{1, 2, 3\}$. Let $\Omega = 12$, $x = (4.5, 5.5, 2)$, $R_1 = R_2$, $p(R_{i=1,2}) = [\underline{x}_{i=1,2} = 7, \overline{x}_{i=1,2} = 8]$, $r_1(x_1) = 9.5$, $r_2(x_2) = 9$, $p(R_3) = [\underline{x}_3 = 1.5, \overline{x}_3 = 5]$. Figure 4 illustrates such representations.

Therefore, we have $\frac{\Omega}{n} = 4 < x_1 = 4.5 < x_2 = 5.5 < \underline{x}_{i=1,2} = 7$, thus for players 1 and 2, $x_{i=1,2} R_{i=1,2} \frac{\Omega}{n}$. For player 3, we have $x_3 = 2, \frac{\Omega}{n} \in p(R_3) = [\underline{x}_3 = 1.5, \overline{x}_3 = 5]$, therefore $x_3 \sim_3 \frac{\Omega}{n}$. Thus, for all $i \in N$, $x_i R_i \frac{\Omega}{n}$ and so $x \in I_{ed}(R)$. Now, for a profile R', suppose that $R_1 = R_2 = R'_1 = R'_2$ for $i = 1, 2$, and for player 3, $p(R'_3) = [\underline{x}'_3 = 2.5, \overline{x}'_3 = 3.5]$, $r'_3(x_3) = 4.25$. Figure 2 illustrates these representations of R'. Thus, for players 1 and 2, it is clear that $LS(x_{i=1,2}, R_{i=1,2}) \cup \{x_{i=1,2}\} \subseteq L(x_{i=1,2}, R'_{i=1,2})$. For player 3, $LS(x_3, R_3) \cup \{x_3\} = [0, \underline{x}_3[\cup\{x_3\}\cup]\overline{x}_3, \Omega] \subseteq L(x_3, R'_3) = [0, x_3] \cup [r'(x_3), \Omega]$. Since $\overline{x}'_3 = 3.5 < \frac{\Omega}{n} = 4 < r'_3(x_3) = 4.25$, then $\frac{\Omega}{n} P'_3 x_3$. Therefore $x \notin I_{ed}(R')$. Q.E.D.

Proposition 12 *The No-Envy correspondence does not satisfy strict monotonicity.*

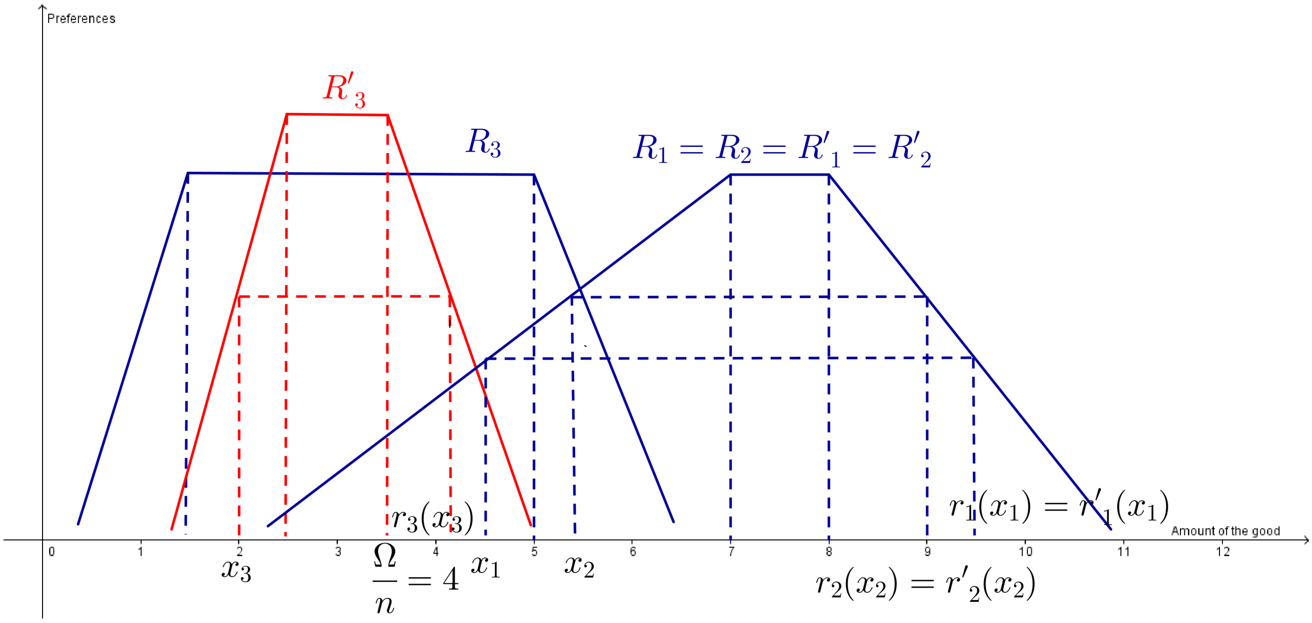

Figure 4: The I_{ed} correspondence does not satisfy strict monotonicity.

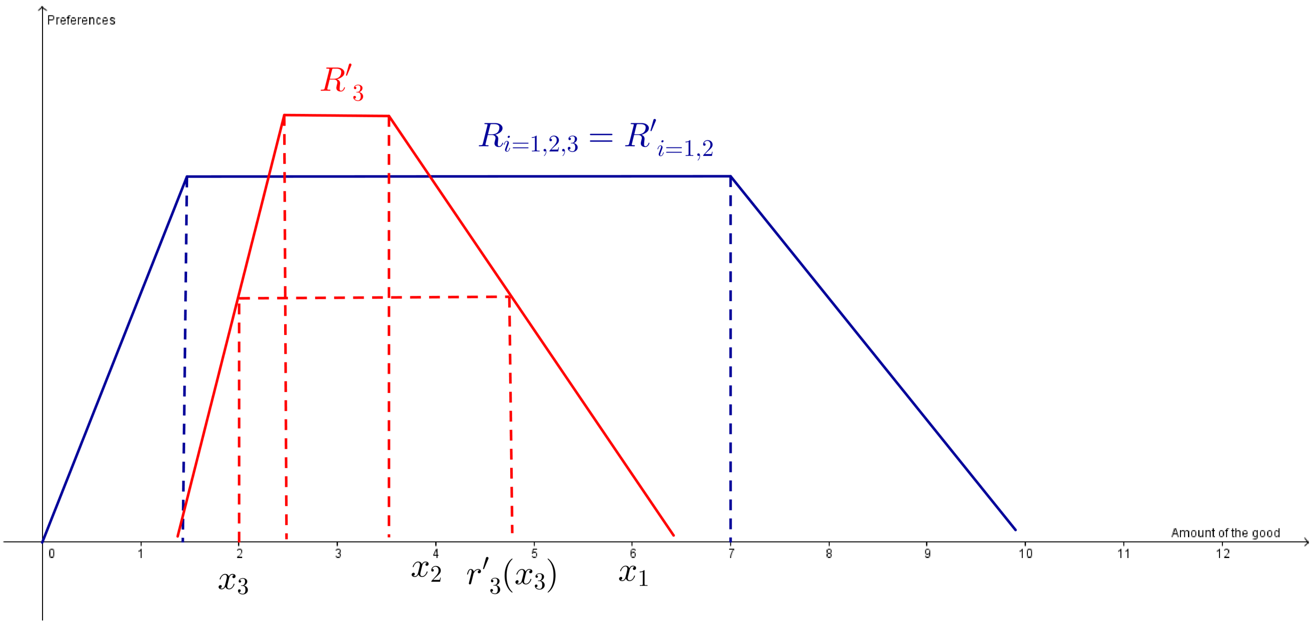

Figure 5: The No-Envy correspondence does not satisfy strict monotonicity.

Proof. Let $R, R' \in \Re_{sp}$ and $N = \{1,2,3\}$. Let $\Omega = 12$, $x = (6,4,2)$, $R_1 = R_2 = R_3$, $p(R_{i=1,2,3}) = [\underline{x}_{i=1,2,3} = 1.5, \overline{x}_{i=1,2,3} = 7]$. Therefore, we have $x \in NE(R)$. Figure 5 illustrates such representations.

Now, for a profile R', suppose that $R_{i=1,2,3} = R'_{i=1,2}$ for $i = 1,2$, and for player 3, $p(R'_3) = [\underline{x}'_3 = 2.5, \overline{x}'_3 = 3.5]$, $r'_3(x_3) = 4.75$. Thus, for players 1 and 2, it is clear that $LS(x_{i=1,2}, R_{i=1,2}) \cup \{x_{i=1,2}\} \subseteq L(x_{i=1,2}, R'_{i=1,2})$. For player 3, $LS(x_3, R_3) \cup \{x_3\} = [0, \underline{x}_3[\cup \{x_3\} \cup]\overline{x}_3, \Omega] \subseteq L(x_3, R'_3) = [0, x_3] \cup [r'(x_3), \Omega]$. Since $\overline{x}'_3 = 3.5 < x_2 = 4 < r'_3(x_3) = 4.75$, then $x_2 P'_3 x_3$. Therefore $x \notin NE(R')$. Q.E.D.

We conclude that the individually rational correspondence from equal division, the no envy correspondence, the $(NE \cap I_{ed})$ correspondence all do not satisfy strict monotonicity. Thus, Proposition 9 does not apply. By applying to Theorem 2, we implement all these correspondences.

6 Conclusion

We have proposed new sufficient conditions, called I-monotonicity and I-weak no-veto power. We have showed that with at least three players, any social choice correspondence satisfying these new conditions together with unanimity can be implemented in Nash equilibria. In the domain of private good economies with single-plateaued preferences, we have proved that, with at least three agents, an SCC is Nash implementable if and only if it satisfies Maskin monotonicity. We have provided positive and negative results for the implementability of some well-Known SCCs.

Finally, we want to return to the impossibility results that we have found, indicating that this impossibility was addressed by Lombardi and Yoshihara (2011), and Doghmi and Ziad (2013) in an area of partial honesty which was developed recently by Matsushima (2008 a, b), and Dutta and Sen (2012).

7 Appendix

Proof. Let $\Gamma = (S, g)$ be a mechanism which is defined as follows: For each $i \in N$, let $S_i = \Re \times A \times \mathbb{N}$, where $\mathbb{N}$ consists of the nonnegative integers. The generic element of strategic space S_i is noted by: $s_i = (R_i, a_i, m_i)$. Each agent announces a preference profile, an optimal alternative for this profile and nonnegative integer. The function g is defined as follows:

Rule 1: If for each $i \in N$, $s_i = (R, a, 1)$ and $a \in F(R)$, then $g(s) = a$.

Rule 2: If for some i, $s_j = (R, a, 1)$ for all $j \neq i$, $a \in F(R)$ and $s_i = (R_i, a_i, m_i) \neq (R, a, 1)$, then:

$$g(s) = \begin{cases} a_i & \text{if} \quad a_i \in LS(a, R_i) \cup I(a, o, R_i) \neq \emptyset \quad \text{for some} \quad o \in LI(a, R_i) \setminus \{a\}, \\ a & \text{otherwise.} \end{cases}$$

Rule 3: In any other situation, $g(s) = a_{i^*}$, where i^* is the index of the player of which the number m_{i^*} is largest. If there are several individuals who check this condition, the smallest index i will be chosen.

Let us show that $F(R) = g(N(g, R, S))$. The proof contains two steps:

Step 1. For all $R \in \Re$, $F(R) \subseteq g(N(g, R, S))$.

Let $R \in \Re$ and $a \in F(R)$. For each $i \in N$, let $s_i = (R, a, 1)$. Then, by rule 1, $g(s) = a$. We want to show that $s \in N(g, R, S))$. Let us choose any individual i and any strategy $\widetilde{s}_i \in S_i$ such that $\widetilde{s}_i = (\widetilde{R}, a_i, \widetilde{m})$. If $a_i \in LS(a, R_i) \cup I(a, o, R_i)$ for some

$o \in LI(a, R_i) \setminus \{a\}$, then, by rule 2, $g(\widetilde{s}_i, s_{-i}) = a_i$. But, since $LS(a, R_i) \cup I(a, o, R_i) \subseteq L(a, R_i)$, then $g(s)R_i g(\widetilde{s}_i, s_{-i})$, thus $s \in N(g, R, S))$. If $a_i \notin LS(a, R_i) \cup I(a, o, R_i)$ for all $o \in LI(a, R_i) \setminus \{a\}$, then $g(s) = g(\widetilde{s}_i, s_{-i})$, thus $s \in N(g, R, S))$.

Step 2. For all $R \in \Re$, $g(N(g, R, S)) \subseteq F(R)$.

Let $s \in N(g, R, S)$. Let us show that $g(s) \in F(R)$. We will study the various possibilities of writing the profile of strategies $s = (s_1, s_2, ..., s_n)$.

Case a: $s = (s_1, s_2, ..., s_n)$. Suppose there exists $(R', a, m) \in \Re \times A \times \mathbb{N}$, with $a \in F(R')$, such that s is defined by $s_i = (R', a, m)$ for any $i \in N$. Then, by rule 1, $g(s) = a$.

Let us take all i and any $b \in LS(a, R'_i) \cup I(a, o, R'_i) \cup \{a\}$ for some $o \in LI(a, R'_i) \setminus \{a\}$. Let $\widetilde{s}_i = (R', b, m')$. Then, by the rule 2, $g(\widetilde{s}_i, s_{-i}) = b$. Since $s \in N(g, R, S)$, $a = g(s)R_i g(\widetilde{s}_i, s_{-i}) = b$. Therefore, $LS(a, R'_i) \cup I(a, o, R'_i) \cup \{a\} \subseteq L(a, R_i)$ for some $o \in LI(a, R'_i) \setminus \{a\}$. By I-monotonicity, $a \in F(R)$.

Case b: $s = (s_1, s_2, ..., s_n)$. Assume there is $i \in N$, $R' \in \Re$ and $a \in A$ such that $a \in F(R')$. For all $j \neq i$, $s_j = (R', a, m)$ and $s_i = (R'_i, a_i, m_i) \neq s_j$, in this case,

$$g(s) = \begin{cases} a_i & \text{if } \ a_i \in LS(a, R'_i) \cup I(a, o, R'_i) \neq \emptyset \ \text{ for some } \ o \in LI(a, R'_i) \setminus \{a\}, \\ a & \text{otherwise.} \end{cases}$$

There are two subcases:

Subcase b_1***:*** If $g(s) = a_i$,
By definition $a_i \in LS(a, R'_i) \cup I(a, o, R'_i) \neq \emptyset$ for some $o \in LI(a, R'_i) \setminus \{a\}$. Take any $b \in LS(a, R'_i) \cup I(a, o, R'_i) \neq \emptyset$ and let $\widetilde{s}_i$ be a deviation by agent i such that $\widetilde{s}_i = (\widetilde{R}, b, \widetilde{m})$. Then, by rule 2, $g(\widetilde{s}_i, s_{-i}) = b$. But, since $s \in N(g, R, S)$, $b \in L(a_i, R_i)$. Hence, we have $a_i \in LS(a, R'_i) \cup I(a, o, R'_i) \subseteq L(a_i, R_i)$ for some $o \in LI(a, R'_i) \setminus \{a\}$. (1)

Next, for any other deviation $j \neq i$ and any $b \in A$, let $\widetilde{s}_j = (\widetilde{R}, b, \widetilde{m})$ a deviation, $\widetilde{m}$ is the unique greatest integer in the profile $(\widetilde{s}_j, s_{-j})$. By rule 2, $g(\widetilde{s}_j, s_{-j}) = b$. Since $s \in N(g, R, S)$, we have $a_i = g(s)R_i g(\widetilde{s}_j, s_{-j}) = b$. Therefore, for all $j \neq i$, $A \subseteq L(a_i, R_j)$. (2)

From (1), (2) and by I-weak no-veto power, we have $a_i \in F(R)$.

Subcase b_2***:*** If $g(s) = a$,
By the same reasoning used in case a, we obtain by I-monotonicity that $a \in F(R)$.

Case c: $s = (s_1, s_2, ..., s_n)$: $\forall i \in N$, $s_i = (R', a, m)$ with $a \notin F(R')$, $g(s) = a$.
Let $b \in A$, $\widetilde{s}_i = (R', b, m')$, where $m' > m$, then, $g(\widetilde{s}_i, s_{-i}) = b$. As $s \in N(g, R, S)$, then, $a = g(s)R_i g(\widetilde{s}_i, s_{-i}) = b$. Therefore, $A \subseteq L(a, R_i)$ for all $i \in N$. By par unanimity, $a \in F(R)$.

Case d: $s = (s_1, s_2, ..., s_n)$: $\exists k_1, k_2, k_3$ where $s_{k_1} \neq s_{k_2}$, $s_{k_1} \neq s_{k_3}$, $s_{k_2} \neq s_{k_3}$, $g(s) = a_l$: m_l is the maximum of the integers m. Let $b \in A$, and $\widetilde{s}_i = (R', b, m_l + 1)$ a deviation. Therefore, $g(\widetilde{s}_i, s_{-i}) = b$. As $s \in N(g, R, S)$, then, $g(s)R_i g(\widetilde{s}_i, s_{-i}) = b$. Thus, $A \subseteq L(g(s), R_i)$ for all $i \in N$. By unanimity, $g(s) \in F(R)$. Q.E.D.

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