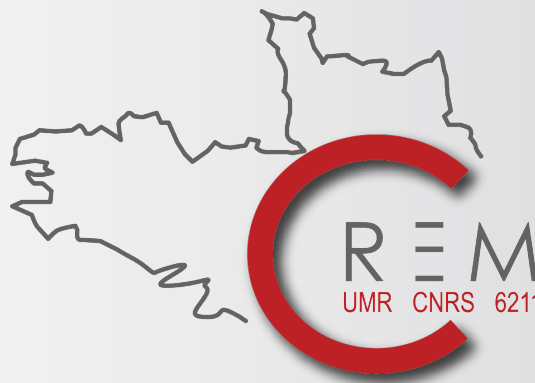




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The College Choice Problem with Priorities

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Abstract

In traditional school choice theory, the assignment mechanisms of students to schools suppose preferences for students and priorities for schools. In this paper, interested in the admission of students to colleges, we assume that all agents have priorities over the members of the opposite side. By considering that students have priorities over colleges, we reduce the incoherence and unfairness of assignments in order to respect the best possible students' educational needs.

Keywords: Matching, Preference, Priority, Coherence, Fairness, Mechanism.

JEL Classification: C78, D03, D47, D63, I20.

1 Introduction

In the last decade, economists have started to study the design of systems used to assign students to schools. School choice is one of the most important subjects in education. It means that parents have the opportunity to choose the school that their children will attend. Traditionally, children are assigned to schools according to their living area. The central issue in a school choice problem is the design of a specific student assignment mechanism, which is a procedure that selects a matching for each school choice problem. In the past, no such explicit mechanism has been provided in the literature or, school choice programs were described by procedures with many disadvantages, like those applied in Boston, Minneapolis and Seattle.

Abdulkadiroğlu and Sönmez proposed in 2003 new procedures to deal with school choice aspects from a mechanism-design perspective. They considered a school choice problem for which there exists a number of students who want to be assigned to a school, with each school having a maximum capacity. In their model, each student has preferences over schools and schools have priorities over students. These priorities, based on state and local laws, can be the obligation to admit students living in a specific geographical zone,

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the obligation to admit students with at least one member of their family being already to the school concerned, and so on. In their pioneering work, they proposed an assignment mechanism, namely the Student Optimal Stable Mechanism (SOSM) with which a matching is selected for a given school choice problem. The former is an adaptation of the widely-studied Deferred Acceptance Algorithm of Gale and Shapley (1962).

In this article, we firstly introduce a mathematically equivalent problem to the school choice one, namely the *college choice problem*, with which we obtain an assignment of students to colleges. Secondly, we answer to a critically important question that is, what happens if students have priorities over colleges. In this perspective, we develop a new problem called the *college choice problem with priorities* where we suppose, on the one hand, that colleges are passive having priorities over students imposed by laws. On the other hand, we consider that students are active having freely determined priorities, that are important criteria for them so as to academically succeed. We call the induced mechanism that selects a matching in this framework, the *Reconsidered Student Optimal Stable Mechanism*. Finally, in our work, we explain the reasons for which considering students' priorities and not their preferences over colleges is crucial in the admission model: we notably analyze the coherence and fairness of any matching.

The remainder is organized as follows. Section 2 presents the college choice problem with priorities. In section 3, we justify the necessity of using students' priorities in the assignment process. Section 4 concludes.

2 The College Choice Problem with Priorities

A college choice problem is mathematically equivalent to a school choice problem, and is defined as a 5-tuple $(N, X, \mathbf{Q}, \mathbf{P}_N^*, \mathbf{P}_X)$, where:

- $N = \{1, \dots, n\}$ is a set of students,
- $X = \{1, \dots, \ell\}$ is a set of colleges,
- $\mathbf{Q} = (Q(1), \dots, Q(\ell))$ is a vector of quotas, with $Q(x)$ being the maximum number of seats available in a college x ,
- $\mathbf{P}_X := \{\mathcal{P}_X : \mathcal{P}_X = \{\mathcal{P}_1, \dots, \mathcal{P}_\ell\}, \mathcal{P}_x \in \mathcal{L}(N)\}$ is the set of all colleges' priority profiles. $\mathcal{L}(N)$ is a set of linear orders on $N \cup \emptyset$, such that $i\mathcal{P}_x j$ means that student i has a strict priority over j for college x . $\emptyset$ denotes the situation of remaining unassigned, and $m(x)$ is the last acceptable alternative for x .
- $\mathbf{P}_N^* := \{\mathcal{P}_N^* : \mathcal{P}_N^* = \{P_1, \dots, P_n\}, P_i \in \mathcal{L}(X)\}$ is the set of all students' preference profiles. $\mathcal{L}(X)$ is a set of linear orders on $X \cup \emptyset$, such that $yP_i z$ means that college y has a strict preference over z for student i . Here, we denote by $\emptyset$ the situation: 'I prefer not to go to a college rather than going to a particular one', and $m(i)$ the last acceptable alternative for i that respects the objective $\tilde{\theta}(i) = \{\text{'succeed my studies for getting a future job'}\}$.

In this framework, a matching between students and colleges is obtained by the student optimal stable mechanism (SOSM) of Abdulkadiroğlu and Sönmez (2003).

In this paper, we are interested in the case where students have priorities and not preferences, a concept too large and encompasses ‘all and anything’. In this perspective, we denote by $C(x)$ the set of all characteristics of a college x , $\forall x \in X$, such that for any student i , $C(x) = (C_{\bar{\theta}(i)}(x), \bar{C}_{\bar{\theta}(i)}(x))$, where:

- $C_{\bar{\theta}(i)}(x) = \{c_{\bar{\theta}(i)}^1(x), \dots, c_{\bar{\theta}(i)}^\sigma(x), \dots, c_{\bar{\theta}(i)}^{\sigma_x}(x)\}$, with $\bar{c}_{\bar{\theta}(i)}^\sigma(x)$ being the σ^{th} characteristic that corresponds to the objective θ , and
- $\bar{C}_{\bar{\theta}(i)}(x) = \{c_{\bar{\theta}(i)}^1(x), \dots, c_{\bar{\theta}(i)}^{\bar{\sigma}}(x), \dots, c_{\bar{\theta}(i)}^{\bar{\sigma}_x}(x)\}$, where $\bar{c}_{\bar{\theta}(i)}^{\bar{\sigma}}(x)$ is the $\bar{\sigma}^{\text{th}}$ characteristic that is not correlated to the objective θ .

Let u be a utility function defined as $u : C \times X \rightarrow \mathbb{R}_+$.

Definition 1. An *educational need* for i , $\forall i \in N$, is a characteristic $c_{\bar{\theta}(i)}(x) \in C_{\bar{\theta}(i)}(x)$ of any college $x \in X$, such that:

$$u(i, c_{\bar{\theta}(i)}^\sigma(x)) > u(i, \bar{c}_{\bar{\theta}(i)}^{\bar{\sigma}}(x)), \forall \bar{c}_{\bar{\theta}(i)}^{\bar{\sigma}}(x) \in \bar{C}_{\bar{\theta}(i)}(x).$$

Definition 2. For all $i \in N$, $\forall x \in X$, a college x *dominates by educational need* a college x' , $\forall x' \in X \setminus \{x\}$, if

$$\sum_{\sigma=1}^{\sigma_x} u_i(c_{\bar{\theta}(i)}^\sigma(x)) > \sum_{\sigma=1}^{\sigma_{x'}} u_i(c_{\bar{\theta}(i)}^\sigma(x')) \quad (1)$$

A priority $\mathcal{P}_i$ is a ranking of acceptable colleges for i that are dominant by educational need. Therefore, we can replace the students’ preferences in a college choice problem by their respective priorities. Thus, we model a new problem, called the *college choice problem with priorities* which is formally defined as a 5-tuple $(N, X, \mathbf{Q}, \mathbf{P}_N, \mathbf{P}_X)$ with:

- $\mathbf{P}_X := \{\mathcal{P}_X : \mathcal{P}_X = \{\mathcal{P}_1, \dots, \mathcal{P}_\ell\}, \mathcal{P}_x \in \mathcal{L}(N)\}$ is the set of all colleges’ priority profiles. $\mathcal{L}(N)$ is a set of linear orders on $N \cup \emptyset$, such that $i\mathcal{P}_x j$ means that student i has a strict priority over j for college x . Here, $\emptyset$ denotes the situation of remaining unassigned.
- $\mathbf{P}_N := \{\mathcal{P}_N : \mathcal{P}_N = \{\mathcal{P}_1, \dots, \mathcal{P}_n\}, \mathcal{P}_i \in \mathcal{L}(X)\}$ is the set of all students’ priority profiles. $\mathcal{L}(X)$ is a set of linear orders on $X \cup \emptyset$, such that $y\mathcal{P}_i z$ means that college y has a strict priority over z for student i . We denote by $\emptyset$ the situation: ‘I prefer not to go to a college rather than go to a particular one’.

In what follows, we use the notation $\#$ to represent the cardinality of a set.

Definition 3. A *matching* of students to colleges is a function $\mu : N \cup X \rightarrow 2^{N \cup X}$ such that:

1. $\mu(i) \subset X$ with $\#\mu(i) \leq 1$, $\forall i \in N$,
2. $\mu(x) \subset N$ with $\#\mu(x) \leq Q(x)$, $\forall x \in X$, and
3. $x \in \mu(i)$ if and only if $i \in \mu(x)$, $\forall i \in N$ and $\forall x \in X$.

In our case, a matching is obtained in same way as by the student optimal stable mechanism (SOSM) of Abdulkadiroğlu and Sönmez (2003). For sake of clarity and without losing integrity, we call the mechanism that selects such a matching, the *Reconsidered Student Optimal Stable Mechanism* (RSOSM), presented below.

Step 1: Each student proposes to his top-ranked college. Each college tentatively assigns its seats to its proposers one at a time following their priority order. Any remaining proposers are rejected.

In general, at

Step k: Rejected students propose to their next best option. Each college considers the students it has been holding together with the new proposers, and tentatively assigns its seats to its proposers one at a time following their priority order. Any remaining proposers are rejected.

The process stops when all students are affected to a college.

Example 1. Let $N = \{1, 2, 3\}$ and $X = \{a, b, c\}$, with a quota equal to the unit for each college. Consider the following profiles of priorities:

$$\begin{array}{ll} \mathcal{P}_1: a\mathcal{P}_1c\mathcal{P}_1b & \mathcal{P}_a: 1\mathcal{P}_a2\mathcal{P}_a3 \\ \mathcal{P}_2: b\mathcal{P}_2a\mathcal{P}_2c & \mathcal{P}_b: 1\mathcal{P}_b3\mathcal{P}_b2 \\ \mathcal{P}_3: a\mathcal{P}_3b\mathcal{P}_3c & \mathcal{P}_c: 2\mathcal{P}_c3\mathcal{P}_c1 \end{array}$$

Initially, all students propose to their top-ranked option. This is, students 1 and 3 make a proposition to college a , while student 2 applies to the college b . Student 2 is accepted by b . Only student 1 is selected by a and 3 is rejected. So, 3 proposes to his second best option i.e. to b , that accepts him. Thus, 2 is rejected. Student 2 now proposes to his second best alternative, which is a . However, college a prefers student 1, so 2 is once again rejected. Student 2 proposes to his last acceptable option, to c , that accepts him. The outcome of the RSOSM is: $\mu(1) = \{a\}$, $\mu(2) = \{c\}$ and $\mu(3) = \{b\}$. $\diamond$

3 Unfairness of Assignments

The aim of this section is to justify the necessity of using the notion of priority for students instead of the large concept of a preference in the assignment process. In fact, when we consider a matching between students and colleges, this may concern both students who try to respect the most possible their educational needs and some others who rather satisfy their own tastes, not necessarily associated to the former. From this, a form of unfairness due to the negligence of the individual educational needs can be detected. We defend here the idea according to which respecting at most students' priorities is crucial as it enables to minimize such an inconvenience. In this perspective, we propose the following conditions.

Relative Coherence (C): If, for any student $i \in N$, a college x dominates another college x' by educational need, $\forall x, x' \in X$, then we say that having a matching μ such

that $\mu(i) = x$ is more *coherent* than having $\mu(i) = x'$. We denote such a situation by $x \succ_C x'$.

Unfairness of Assignment (UA): $\forall i, j \in N$, when, $\forall x, x' \in X$,

$$P_i = P_j : x P_j x' | \mathcal{P}_i : x \mathcal{P}_i x', \mathcal{P}_j : x' \mathcal{P}_j x$$

and $\varphi_i(x) = \varphi_j(x)$, then we obtain $\mu(i) = x'$ and $\mu(j) = x$, where $\varphi_i(x)$ is the level of outcome φ of i (scores, talent, etc.) necessary to access to college x .

In this case, we deduce that student i obtains an unfair assignment. Interested until now in the problem of fairness, we turn our attention to the problem of efficiency.

N-Pareto dominance (N-PAR): For any matching μ and μ' , we say that,

$$\mu \succ_{N-PAR} \mu' \text{ if } \frac{n(\mu)}{n} > \frac{n(\mu')}{n}$$

with,

$$\begin{cases} n(\mu) = \#\{i \in N : \mu(i) P_i \mu'(i)\} & \text{if we consider an SOSM} \\ n(\mu) = \#\{i \in N : \mu(i) \mathcal{P}_i \mu'(i)\} & \text{if we consider an RSOSM} \end{cases}$$

Therefore, we deduce the following condition:

N-Pareto Efficiency (NPE): A matching μ is *N-Pareto Efficient* if there does not exist a matching μ' such that $\mu' \succ_{N-PAR} \mu$ given that $\frac{n(\mu)}{n} = 1$.

Relatively to the (NPE), we now turn our attention to the proportion of students for whom their respective assignment is fair. In this perspective, we announce the next conditions.

Fair Pareto dominance (F-PAR): For any matching μ and μ' we have,

$$\mu \succ_{F-PAR} \mu' \text{ if } \frac{1}{n} \#\{i \in N : \mu(i) \mathcal{P}_i \mu'(i)\} > \frac{1}{n} \#\{i \in N : \mu'(i) \mathcal{P}_i \mu(i)\}$$

Fair Pareto Efficiency (FPE): A matching μ is *Fairly Pareto Efficient* if there does not exist a matching μ' such that $\mu' \succ_{F-PAR} \mu$,

$$\#\{i \in N : \mu(i) \mathcal{P}_i \mu'(i)\} = n$$

After having introduced some basic conditions such as the coherence and the fairness, we can now proceed to the main objective of our research study, that is to defend the use of priorities instead of preferences in an assignment process. For this purpose, we propose the following theorem that analyzes the relation of dominance between any matching obtained with the SOSM (μ^{SOSM}) and the RSOSM (μ^{RSOSM}).

Theorem 1. For any matching $\mu = \mu^{SOSM}$ and $\mu' = \mu^{RSOSM}$, we have:

- (a.) $\mu^{RSOSM} \succ_C \mu^{SOSM}$
- (b.) $\mu^{RSOSM} \succ_{F-PAR} \mu^{SOSM}$

Proof. Taking a population $N = \{1, 2, 3, 4, 5\}$ and $X = \{a, b, c, d, e\}$, we consider the following assignments, for a quota equal to the unit for each college: $\mu^{\text{SOSM}}(1) = a$, $\mu^{\text{SOSM}}(2) = b$, $\mu^{\text{SOSM}}(3) = c$, $\mu^{\text{SOSM}}(4) = d$, $\mu^{\text{SOSM}}(5) = e$, according to the following students' preferences on colleges:

P_1 : $cP_1dP_1\boxed{a}$
 P_2 : $\boxed{b}P_2aP_2eP_2c$
 P_3 : $\boxed{c}P_3bP_3aP_3d$
 P_4 : $cP_4bP_4\boxed{d}$
 P_5 : $cP_5aP_5bP_5\boxed{e}$

Assume the students' priorities on acceptable colleges, according to their educational needs:

$\mathcal{P}_1$: $d\mathcal{P}_1c\mathcal{P}_1\boxed{a}$
 $\mathcal{P}_2$: $e\mathcal{P}_2a\mathcal{P}_2c\mathcal{P}_2\boxed{b}$
 $\mathcal{P}_3$: $a\mathcal{P}_3b\mathcal{P}_3\boxed{c}\mathcal{P}_3d$
 $\mathcal{P}_4$: $b\mathcal{P}_4c\mathcal{P}_4\boxed{d}$
 $\mathcal{P}_5$: $c\mathcal{P}_5a\mathcal{P}_5\boxed{e}\mathcal{P}_5b$

Students are supposed to have the same talent in order to have the same opportunity to access to colleges.

(a.) The above situation is incoherent. Indeed, respecting students' priorities: student 1 should be assigned to the college c or d (in the best case). Student 2 should be assigned to the college c , a or e (in the best case). Student 3 should be assigned to the college b or a (in the best case). Student 4 should be assigned to the college c or b (in the best case). Student 5 should be assigned to a or c (in the best case).

(b.) Moreover, the assignments μ^{SOSM} previously provided are unfair: students 1, 4 and 5 obtained a (relative) less acceptable matching which does not correspond to their top-ranking in $\mathcal{P}$. Next to that, students 2 and 3 obtained their top-ranking in P that does not correspond to their top-ranking in $\mathcal{P}$.

Suppose now that, according to students' priorities, we have $\mu^{\text{RSOSM}}(1) = d$, $\mu^{\text{RSOSM}}(2) = e$, $\mu^{\text{RSOSM}}(3) = b$, $\mu^{\text{RSOSM}}(4) = c$, $\mu^{\text{RSOSM}}(5) = a$. It is obvious that μ^{RSOSM} is less unfair and relatively coherent.

According to Abdulkadiroğlu and Sönmez (2003), any matching μ^{SOSM} is not always Pareto Efficient. In our framework, this is equivalent to say that it is not N-Pareto Efficient. Thus, as any matching μ^{RSOSM} is obtained in the same way as a matching μ^{SOSM} (see Section 2), the former is not necessarily N-Pareto Efficient. Nevertheless, in our example, not only $\mu^{\text{RSOSM}} \succ_{F\text{-}PAR} \mu^{\text{SOSM}}$ (over the RSOSM, all students have a fairly improved assignment that respects the most their respective priorities), but we also have the extreme case where μ^{RSOSM} is F-Pareto Efficient (assuming that there does not exist any other matching that fairly dominates it!). $\square$

4 Conclusion

The main contribution of this paper is to develop an assignment mechanism of students to colleges in which all agents have priorities over the members of the opposite side. In this perspective, we introduce a new problem, namely the 'college choice problem with

priorities'. We call the induced mechanism that selects a matching for such a problem, the 'reconsidered student optimal stable mechanism', which is an adaptation of the student optimal stable mechanism of Abdulkadiroğlu and Sönmez.

As our mechanism is notably based on freely determined students' priorities, it reduces unfairness between them. To our knowledge, there is no existing framework that provides insights on this subject in the theory of matching. However, the notion of priority is important and has always to be taken into account so as to obtain fair students' assignments that respect the best possible their educational needs.

After all, this is the magic of mechanism design! Parts of a given mechanism can be changed in order to achieve a goal or a goal more effectively. Our purpose is to avoid the inconvenience of lack of fairness between students when they want to enter a college, study and succeed so as to reach their ambitions.

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