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Optimal contract with private information on cost expectation and variability

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Abstract

A multidimensional-and-sequential screening problem arises in a framework where the agent is privately informed about expected cost and cost variability and, subsequently, learns the realized cost as well. As the principal's marginal surplus function becomes less concave/more convex, the optimal mechanism reflects progressively stronger incentives to mimic less inefficient types, and to misrepresent the cost variability relative to the expected cost. When the principal's knowledge imperfection about the cost variability is sufficiently less important than that about the expected cost, quantities are pooled with respect to the former for a high-expected-cost agent. A low-expected-cost agent is not assigned the first-best output at least in some state of nature.

Keywords: Multidimensional screening; Sequential screening; Expected cost; Cost variability; Marginal surplus function

J.E.L. Classification Numbers: D82

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1 Introduction

Before undertaking new activities in uncertain environments, whether to be run for public or private purposes, firms typically develop a feasibility analysis that is meant to assess prospective costs. This analysis consists in determining not only the expected value of the cost, but also its variability, which measures the uncertainty associated with the activity.[1] The outcome that is obtained is not necessarily publicly observable. Therefore, when firms perform the analysis with regards to a delegated activity, they are likely to have an information advantage about the two assessed cost components *vis-à-vis* the delegating party.

Information problems of this kind may plague any contractual relationship in which some good or service is procured from an outside supplier under uncertainty about production costs. In particular, they are found to be relevant for governmental agencies (regulators, public authorities, and other institutions) dealing with firms (regulated monopolies, franchisees) for the execution of activities of general interest. As an illustration, in transportation projects, concerns typically arise about overoptimistic estimates of both expected values and error forecasts. Not only this may reflect the presence of technical fallacies. It may also follow from the strategic manipulation of the truly estimated expected value and degree of uncertainty, unless incentives to behave opportunistically are contractually removed.[2]

Although it is natural that, in uncertain environments, agents hold private information about both the expected value of some parameter that matters in the relationship with the principal and its variability, the literature has not yet studied how principals should design screening mechanisms to properly address information issues of this kind. Riordan and Sappington [20] and Spulber [22] focus on situations in which, at the outset of the relationship with the principal, the agent is privately informed about the sole expected cost of production. Miravete [19] - [18] and Courty and Li [9] take a similar approach, though in a different context. They tackle the issue of private knowledge about the expected consumption benefit, looking at situations in which the agent is a consumer who purchases a product from a monopolist. Courty and Li [9] also analyze the case in which the consumer observes the variability of the perspective consumption benefit but not its expected value.

In this article, we characterize the optimal incentive mechanism for a production activity that a principal delegates to an agent who is privately informed about both the expected value and the variability of the unit cost of production at the contracting stage. In line with previous works, we make the study truly positive by focusing on situations in which, at a later stage, the agent observes privately also the realized cost, which can be either low (the good state) or high (the bad state) *i.e.*, we allow for sequential learning on the agent's side. Hence, overall, the agent holds two pieces of private information (jointly representing the first-

[1] A feasibility analysis can, of course, be required to assess prospective benefits as well. In this study, however, attention is restricted to costs.

[2] Several studies about transportation projects provide evidence that, in the latter, costs and errors about expected costs turn out to be systematically bigger than originally estimated. It is argued that wrong predictions are largely due to firms' strategic behaviour (see, for instance, Flyvbjerg [11], Flyvbjerg *et alii* [12] and Flyvbjerg *et alii* [13]).

stage two-dimensional type) at the outset of the relationship with the principal, and learns an additional piece of information (the actual state of nature) once uncertainty vanishes. By representing this information structure, we bridge the strand of literature on sequential learning problems (that we mentioned above) with the studies on multidimensional information issues that explore situations in which, at the contracting stage, the agent holds more than one piece of information, related either to one activity (Armstrong [1], Asker and Cantillon [5]) or to two activities (Dana [10], Armstrong and Rochet [2]), and there is nothing to learn over time.

Given the information structure described above, as in Riordan and Sappington [20] and in Courty and Li [9], the principal relies on a sequential screening mechanism, under which the agent is required to disclose information twice *i.e.*, when parties sit at the contracting table and when uncertainty vanishes. Specifically, the mechanism includes a prior menu of optional schedules, among which the agent chooses by reporting the first-stage two-dimensional type, followed by a menu of specific contractual options, from which the agent draws the final policy by reporting the realization of the cost. The focus on a mechanism with this structure is motivated by that, as frequent in incentive problems, it is desirable for the principal to collect a report every time the agent acquires some new piece of information, rather than to ignore it, prior to the policy choice.

Within this framework, we address a number of specific issues. First, we attempt to understand which features of the economic environment where the principal/agent relationship unfolds drive the optimal policy design, and how exactly they affect that policy. A major characteristics of multidimensional screening problems is that, depending upon the features of the environment in which activities are run, different combinations of relevant incentive constraints of the agent come to matter, hence different contractual policies are optimally designed. To pin down all possible solutions to the problem that the principal faces in our model, it is essential to identify the features that are relevant in the particular setting that we consider, in which a non-negligible complication follows from the fact that information is elicited not only on more than one dimension, but also sequentially. Once all solutions are characterized, we ask two further questions. What could one learn out of those solutions in terms of efficiency/rent-extraction trade-off, efficiency being expressed in expected terms in the considered setting? Which are the very consequences of the screening problem being both multidimensional and sequential?

Our first result is that the optimal multidimensional-and-sequential screening mechanism depends finely upon (i) the shape of the marginal surplus function, which expresses the preferences of the principal for the good produced by the agent, and (ii) the "spread index," which measures the importance of the principal's knowledge imperfection about the expected value of the cost, relative to that about the variability of the cost. The combined relevance of the principal's preferences, on one side, and the relative importance of the principal's gap about the two information pieces, on the other, has not appeared to be substantive in the information problems that the theory of incentives has explored so far. We hereafter illustrate why these two elements matter jointly in our setting.

For the agent to release information about the realized unit cost at the second stage of the relationship, the principal must set the production level in the good state at least as large as in the bad state for each first-stage type. Absent any requirement on the magnitude of the agent's *ex post* payoff,[3] this is the only restriction that second-stage screening imposes on first-stage decisions. At the contracting stage, the principal chooses the quantities to be produced at the second stage, trading off expected efficiency against rent-extraction purposes. Peculiar to our framework is that rents depend upon expected quantities and expected quantity differences, not directly upon production levels. Under this circumstance, efficiency concerns involve both the expected surplus over the quantities to be provided in the two states of nature and the expected surplus difference between those same quantities. Because of this, the loss associated with the quantity distortions is finely related to the curvature of the marginal surplus function. As the latter becomes less concave/more convex, different contractual solutions emerge, at optimum, reflecting different combinations of binding incentive constraints.

Let us now come to the spread index, the second relevant element in the characterization of the optimal mechanism. The reason why it matters is that not only, as usual, the rent-extraction benefits that the principal obtains through quantity distortions depend upon how costly each piece of information is *per se*. They also depend upon how costly each piece of information is relative to the other. This all works as follows. Any raise in the bad-state production level triggers opposite effects in terms of incentives to misrepresent expected cost and cost variability. The way in which the bad-state production is fixed in the optimal mechanism, relative to the good-state production, captures the need to compromise these two contrasting effects. Therefore, the optimal screening mechanism displays different features not only for different shapes of the marginal surplus function, as already explained, but also for different values of the spread index.

The characterization of the whole set of solutions allows us to draw insights about the way in which relevant incentives evolve as the marginal surplus function becomes less concave/more convex and the contractual policy is changed consequently. We find that, given the value of the spread index, information rents and production levels in the optimal mechanism reflect progressively stronger incentives to mimic less inefficient types. That is, less inefficient types become more and more likely to be announced. Accordingly, more surplus is given up to the agent in order to elicit information. This is because, the marginal surplus function being less concave/more convex, the principal is more concerned with efficient production. That is, she cares about containing distortions, and particularly privileges low-cost output (*i.e.*, output produced by less inefficient types) with respect to high-cost output (*i.e.*, output produced by more inefficient types).[4] In return for overall more efficient production, she then grants higher information rents.

Besides, as the marginal surplus function becomes less concave/more convex, information

[3] The issue of the agent's *ex-post* liability is considered at a later stage.

[4] Throughout the paper the feminine pronoun *she* is referred to the principal, the masculine pronoun *he* to the agent, the neuter pronoun *it* to the type.

rents and production levels reflect progressively stronger incentives to misrepresent the variability of the cost relative to the expected value of the cost. Again, this is to be understood by considering the stronger concern of the principal for efficient production that we mentioned above. On top of privileging production by less inefficient types, as we said, this concern is addressed by keeping output large in the good state relative to the bad state, for any given type. Then, the rents designed to prevent misreporting on cost variability are set high, with the obvious consequence that this lie becomes more attractive relative to a lie on the expected cost.

Importantly, the various solutions that the screening problem attains in our framework, depending upon the shape of the principal's marginal surplus function, would reduce to one single solution, not coinciding with any of those that we pin down, if the problem were not sequential. This emerges from the comparison with purely multidimensional screening problems, in which the agent runs two activities and holds one piece of private information on each such activity (Armstrong and Rochet [2], for instance). In those problems, the quantity chosen for the first activity is optimally conditioned also on the information concerning the second activity if and only if the two pieces of information are correlated. Absent correlation, the principal faces twice a replica of a standard unidimensional adverse selection problem, and each quantity reflects information on the activity to which it pertains. In our model, optimal quantities depend upon both expected cost and cost variability, despite these not being correlated, because the second-stage state of nature is, in fact, an aggregate resulting from these two pieces of first-stage information.

These insights are drawn for any possible size of the spread index. Nonetheless, the determination of the whole set of solutions allows us to further highlight an interesting aspect of the optimal mechanism, related to the exact magnitude of the spread index. As the latter increases, meaning that the knowledge imperfection about the expected cost becomes progressively more important, relative to that about the cost variability, similarities emerge with the mechanisms that Riordan and Sappington [20] and Courty and Li [9] characterize for the case in which, at the contracting stage, the agent is privately informed about the sole expected value of the parameter that is relevant in the relationship with the principal. Nonetheless, the multidimensional nature of the incentive problem that we explore involves that similarities are only partial. Specifically, one first similarity resides in that the principal assigns to an agent with high expected cost some given production level, whatever his cost variability. That is, she induces pooling along the second dimension of private information. However, this does not mean that the principal always needs to standardize the production rule with respect to cost variability when the spread index is large. Actually, the optimal output profile is fully separating even in this case, provided the forecast about the agent's cost is optimistic. Another similarity consists in that a low-expected-cost agent may be required to produce the first-best quantities, at least in some state of nature. Yet, unlike in purely sequential frameworks, this does not always occur in our setting. Again depending upon the preferences of the principal for the good and the relative importance of the two information problems, distortions may also be induced in the low-expected-cost output. Specifically, this occurs in the case of very concave marginal surplus and very low spread index. The mechanism that we pin down naturally collapses onto a purely sequential mechanism only in the limit case of commonly known cost variability, in which the solution to the screening problem is no longer differentiated according to the shape of the principal's marginal surplus function.

Overall, the comparison between the mechanism that proves optimal in our model and those that are found to be optimal in purely multidimensional or sequential frameworks makes it clear that the importance that the features of the surplus function have in determining its structure is a very consequence of the problem being both multidimensional and sequential.

The results presented so far are derived in a full-commitment framework in which the principal is able to keep the agent in the contract by simply ensuring that he breaks even in expectation. One might object that this circumstance is seldom verified, in practice, and that it would be more realistic and, thus, more useful to look at situations where the agent is protected by limited liability. In fact, introducing limited liability on the agent's side would bring about analytical complications without affecting the qualitative nature of results. To evidence this, as a final step of the work, we admit *ex-post* participation constraints and, referring to the case of linear marginal surplus for illustrative purposes, we show that our original analysis is not restrictive. Actually, the information problem comes out to be similar to the one that the principal faces, under the same circumstances, in the absence of *ex-post* participation constraints. Hence, the optimal mechanism maintains its main features in the new environment. With regards to purely sequential problems, Krahmer and Strausz [15] find that, when *ex-post* participation constraints are imposed, the optimal contract is a static one, and exclusively refers to the agent's aggregate final information. This is not the case in our setting. Indeed, because the second-stage state is linked directly (rather than stochastically) to the first-stage type, the principal does benefit from requiring the agent to report information twice, rather than only after he has observed the final cost, even in the presence of *ex-post* participation constraints.

The reminder of the article is organized as follows. After reviewing the mainly related literature hereafter, we present the analytical framework and set the programme of the principal in Section 2. In Section 3, we develop some initial steps of analysis, and provide a few preliminary results. The optimal screening mechanism is characterized in Section 4 and discussed in Section 5, where the core lesson of the study is conveyed. Section 6 briefly concludes. Mathematical details are relegated to the Appendix.

1.1 Mainly related literature

In a early work on agency relationships with information evolution over time, Riordan and Sappington [20] explore the problem of a regulator who auctions out a franchise public-service contract to a firm that holds private information about the expected cost at the tendering stage and privately observes the realized cost at a later stage.[5] Still in an auction model, Spulber [22] represents the situation in which, at the tendering stage, each participant knows privately the possible levels of a cost overrun that he may incur at a later stage, whereas the basic cost is commonly observed. This is tantamount to having the agent privately informed about either the cost expectation or its variability, with perfect correlation between the two. Hence, private information at the tendering stage is one-dimensional, as in the model of Riordan and Sappington [20]. Miravete [19] - [18] and Courty and Li [9] study an information problem analogous to that of Riordan and Sappington [20], in a different setting. Specifically, they consider a monopolist selling a product to a consumer who knows privately his expected benefit from consumption at the initial stage and observes privately his actual preference for the product at a later stage. Additionally, Courty and Li [9] investigate the alternative situation in which the customer is initially informed about his taste variability, rather than its expected value.[6] None of these authors considers the possibility that the agent hold two-dimensional private information (including both the expectation and the variability of the relevant cost/benefit) at the time he signs the contract with the principal, whereas we do so with regards to the cost of the agent's activity. The focus on a sequential mechanism, which we share with Riordan and Sappington [20] and Courty and Li [9], allows us to account for the principal's wish to receive a report every time the agent learns something privately, as we mentioned. From this standpoint, we rather diverge from Spulber [22] and Miravete [19] - [18]. In the former model, second-stage information disclosure is unfeasible because contracts are not enforceable; in the latter, sequential screening is neglected as the goal is to compare two mechanisms that are based on either first-stage or second-stage information release only.

Among the studies on multidimensional screening, Dana [10] and Armstrong and Rochet [2] assume that the agent executes two activities for the principal, holding private information about the cost of either activity. In this setting, the production level of each activity is used as a screening instrument for the corresponding piece of private information. In Asker and Cantillon [5], the agent runs a single business and knows privately both the operating and the fixed cost. As the latter is not related to the produced quantity, the principal has one sole screening device related to the production level. Similarly, in Armstrong [1] the agent executes a single activity but because his two pieces of information (namely, production cost and product demand) are both related to the production level, the principal uses the (sole)

[5] A few more recent studies extend the analysis of Riordan and Sappington [20] to environments in which either the contract includes such additional factors as product quality (Che [8]; see also Che [7] for an overview) or it accounts for various sources of information asymmetries (Armstrong and Sappington [3]) or both (Asker and Cantillon [5]). However, these papers do not tackle problems related to information learning on the agent's side. The information structure they consider is tantamount to having perfectly correlated first- and second-stage information in the model of Riordan and Sappington [20] (on this point compare footnote 5 in Che [8]).

[6] In details, Courty and Li [9] consider two types of customers having a continuum of possible valuations of the product and study the following two situations: (*i*) one type first-order stochastically dominates the other; (*ii*) one type faces greater valuation uncertainty than the other in the sense of mean-preserving spread. The first case corresponds to having private information about the sole cost expectation, the second case to having private information about the sole cost variability.

product price to screen both. In our model, as in Asker and Cantillon [5] and in Armstrong [1], the agent is delegated a single activity. Nonetheless, unlike in those papers, two quantity-related screening devices, namely the expected production and the expected difference between production levels, come up to be available for the principal at the stage in which she faces the two-dimensional information problem. For this reason, despite the difference in the number of activities that the agent executes for the principal, the results derived in our framework compare more naturally with those of Armstrong and Rochet [2], which explains why we refer systematically to the latter throughout the analysis.

2 The model

We consider the relationship between a principal (P) and an agent, both risk-neutral, for the production of q units of some good at a payment t.[7] The expected unit cost of production θ is drawn from the set $\{\theta_L, \theta_H\}$, $\theta_H > \theta_L > 0$, with commonly known probabilities ν and $1-\nu$, respectively. We denote $\Delta\theta = \theta_H - \theta_L$. The unit cost is realized after the contract is signed and before production takes place. It can be either $\theta - \sigma$ or $\theta + \sigma$ with equal probabilities. The former (low cost) represents the "good" state resulting from a positive shock, the latter (high cost) the "bad" state resulting from a negative shock. By attaching equal probabilities to these two events, we prevent the otherwise asymmetric distribution of high and low unit costs from imposing structure on the optimal mechanism. The parameter σ, expressing the uncertainty about the unit cost realization (*i.e.*, the unit cost variability), is drawn from the set $\{\sigma_L, \sigma_H\}$, $\sigma_H > \sigma_L > 0$, with commonly known probabilities μ and $1-\mu$, respectively. We also denote $\Delta\sigma = \sigma_H - \sigma_L$. Throughout the article, we refer to the generic realization of the two cost parameters as to θ_i and σ_j, with $i, j \in \{L, H\}$. To simplify the analysis, we take θ_L, θ_H, σ_L and σ_H to be such that $\Delta\theta > \Delta\sigma$. In words, we suppose that the knowledge imperfection about the expected cost is more important than that about the cost variability. This means that a higher expected cost corresponds to a higher true cost, even when it is associated with a higher variability and the good state is realized (*i.e.*, $\theta_H - \sigma_H > \theta_L - \sigma_L$).

Information structure Before sitting at the contracting table, the agent observes privately both the expected cost θ_i and the cost variability σ_j, $i, j \in \{L, H\}$. Hence, when the contract is signed (the *first stage* of the relationship), he enjoys a double information advantage. We denote ij the agent's type for any realized pair (θ_i, σ_j) and $\Upsilon \equiv \{LL, LH, HL, HH\}$ the set of feasible types. The agent acquires a new information advantage when the state of nature is determined (the *second stage* of the relationship). Indeed, he learns privately whether the actual unit cost is $\theta_i - \sigma_j$ or $\theta_i + \sigma_j$. An important aspect is that the (second-stage) state of nature depends directly upon the (first-stage) type of the agent.

[7] As far as a public project or an activity of general interest is concerned, the agent can be viewed as a contractor, a supplier, a regulated (possibly local) monopoly; the principal as a governmental agency, a regulator etc.

Payoffs under symmetric information and first-best allocation For each type $ij \in \Upsilon$, we let $(\underline{q}_{ij}, \underline{t}_{ij})$ denote the allocation to be implemented when the cost is $\theta_i - \sigma_j$, and $(\overline{q}_{ij}, \overline{t}_{ij})$ that to be implemented when the cost is $\theta_i + \sigma_j$. Accordingly, under symmetric information, the profits of the agent are given by

$$\underline{\pi}_{ij} = \underline{t}_{ij} - (\theta_i - \sigma_j)\,\underline{q}_{ij} \tag{1}$$

$$\overline{\pi}_{ij} = \overline{t}_{ij} - (\theta_i + \sigma_j)\,\overline{q}_{ij}. \tag{2}$$

Assuming that there is no discount factor and recalling that good and bad state occur with equal probability, the payoff of the agent is written as

$$\Pi_{ij} = \frac{1}{2}(\underline{\pi}_{ij} + \overline{\pi}_{ij}). \tag{3}$$

Production of q units of the good yields the gross surplus $S(q)$ to P. Under symmetric information, the net benefit of P is given by

$$V_{ij} = \frac{1}{2}\left[(S(\underline{q}_{ij}) - \underline{t}_{ij}) + (S(\overline{q}_{ij}) - \overline{t}_{ij})\right]. \tag{4}$$

We take the surplus function to be three-time differentiable, with $S'(q) \geq 0$ and $S''(q) \leq 0$, $\forall q \geq 0$. We further take $S'''(q)$ to maintain the same sign for all values of q for which $S'(q) > 0$. This involves that there is a first range of values of q, namely $[0, \widetilde{q})$, for some $\widetilde{q} > 0$, over which $S'(q)$ is strictly decreasing and, if $\widetilde{q} < +\infty$, there is also a second range $[\widetilde{q}, +\infty)$ over which $S'(q)$ is constant and equal to zero.[8] We also assume that $S'(0)$ is finite but sufficiently large to ensure that the optimal quantity is positive. Moreover, $\lim_{q \to +\infty} S'(q) = 0$ so that the optimal quantity is finite. Taken altogether, the previous assumptions warrant that the problem of P (to be presented below) has an interior solution and that, more interestingly, the analysis of the optimal contract can be developed for different shapes of $S'(q)$. This is going to be a core aspect of our work.[9]

At first best (FB hereafter), an agent of type $ij \in \Upsilon$ is required to produce quantities such that $S'(\underline{q}^*_{ij}) = \theta_i - \sigma_j$ and $S'(\overline{q}^*_{ij}) = \theta_i + \sigma_j$, and is given up no rent *i.e.*, $\Pi^*_{ij} = 0$.

Timing The game unfolds as follows. Nature draws θ_i and σ_j, the agent observes them privately and then meets P at the contracting table. At the first stage of the relationship, P offers to the agent the truthful menu of optional contracts $\{(\underline{q}_{ij}, \underline{t}_{ij}); (\overline{q}_{ij}, \overline{t}_{ij})\}$, $\forall ij \in \Upsilon$. The agent reports ij to P and the contract targeted to type ij is signed. Both parties fully commit

[8] For instance, if P is a regulator/governmental agency and $S(q)$ reflects consumer preferences over the good q, the case of $S' = 0$ represents the situation in which quantity is sufficiently large that consumers are no longer willing to pay for extra units of the concerned good.

[9] As $S'''(q)$ is assumed to have a constant sign, if we were to take $S'(q) > 0$ and $S''(q) < 0$ for all $q \geq 0$, then the only meaningful case would be that of $S'(q)$ strictly convex for all non-negative values of q (see Menegatti [17] for a proof). The same would occur under the assumption that $\lim_{q \to 0} S'(q) = +\infty$. In either case, the solutions presented in the sequel of the work would reduce to those that arise for $S'(q)$ strictly convex.

to this contract.[10] At the second stage, the agent observes privately whether a positive or a negative shock has affected the cost (hence, whether the realized state is good $(\theta_i - \sigma_j)$ or bad $(\theta_i + \sigma_j))$ and announces it to P. Accordingly, out of the stipulated contract, either the allocation $(\underline{q}_{ij}, \underline{t}_{ij})$ or the allocation $(\overline{q}_{ij}, \overline{t}_{ij})$ is effected. Remarkable aspects of this game are that P collects a report every time the agent holds/acquires some piece of private information, and that the first-stage report is two-dimensional. It means that P engages in both *sequential* and *multidimensional* screening. We shall elaborate further on this aspect after presenting the programme of P.

2.1 The programme of the principal

Under asymmetric information, the Revelation Principle applies, and P can restrict attention to direct mechanisms that induce truthtelling. The optimal mechanism is pinned down by solving the following programme:

$$\underset{(\underline{q}_{ij},\underline{t}_{ij});(\overline{q}_{ij},\overline{t}_{ij})}{Max} \ \frac{1}{2}\sum_{ij\in\Upsilon} E_{ij}\left[(S(\underline{q}_{ij}) - \underline{t}_{ij}) + (S(\overline{q}_{ij}) - \overline{t}_{ij})\right]$$

$$\text{subject to} \qquad (\Gamma)$$

$$\begin{cases} \Pi_{ij} \geq \frac{1}{2}\left\{\left[\overline{t}_{i'j'} - (\theta_i+\sigma_j)\,\overline{q}_{i'j'}\right] + \left[\underline{t}_{i'j'} - (\theta_i-\sigma_j)\,\underline{q}_{i'j'}\right]\right\},\ \forall ij, i'j' \in \Upsilon & (IC_{ij}^{i'j'}) \\ \Pi_{ij} \geq 0,\ \forall ij \in \Upsilon & (PC_{ij}) \\ \underline{\pi}_{ij} \geq \overline{\pi}_{ij} + 2\sigma_j \overline{q}_{ij},\ \forall ij \in \Upsilon & (\underline{ic}_{ij}) \\ \overline{\pi}_{ij} \geq \underline{\pi}_{ij} - 2\sigma_j \underline{q}_{ij},\ \forall ij \in \Upsilon & (\overline{ic}_{ij}) \end{cases}$$

The purpose of P is to maximize the expected surplus, net of the payment to be made to the agent for the production of the good, subject to a number of constraints. The first-stage incentive constraint $(IC_{ij}^{i'j'})$ warrants that type $ij \in \Upsilon$ prefers declaring the truth rather than delivering any other report $i'j' \in \Upsilon$. Additionally, the first-stage participation constraint (PC_{ij}) requires that type ij be guaranteed a non-negative payoff. Lastly, the second-stage incentive constraints $(\underline{ic}_{ij})$ and $\left(\overline{ic}_{ij}\right)$ warrant that type ij correctly announces whether a positive or a negative shock, respectively, has occurred.[11]

[10] One can think of commitment by the agent as ensured by imposing cancellation fees at the production stage. Alternatively, *ex-post* participation constraints could be introduced, a possibility that will be considered in Section 6. In turn, one can think of commitment by P as ensured by the presence of legislation and procedural requirements that allow her to credibly engage in enacting the agreed policies in the future. For a discussion on commitment in continuing relationships, see, for instance, Baron and Besanko [6].

[11] For each type $ij \in \Upsilon$, on top of $(IC_{ij}^{i'j'})$, an additional first-stage incentive constraint is to be considered. This is the constraint whereby the $ij-$agent not be tempted to declare $i'j'$, knowing that he will also cheat on the realized shock at the second stage. In Γ these constraints are omitted as they are implied by $(IC_{ij}^{i'j'})$, $(\underline{ic}_{ij})$ and $(\overline{ic}_{ij})$.

3 Preliminary analysis

Prior to characterizing the optimal multidimensional-and-sequential screening mechanism, we develop a few preliminary steps of analysis that illustrate how one can identify the constraints that are relevant in Γ. Ultimately, these constraints determine the optimal rents and the distortions to be possibly induced in the production levels.

3.1 Relevant constraints and type ranking

We begin by stating a standard result concerning the second stage of the contractual relationship.

Lemma 1 *At the solution to* Γ :

$$\underline{q}_{ij} \geq \overline{q}_{ij}, \ \forall ij \in \Upsilon. \tag{5}$$

For any pair $\left\{\underline{q}_{ij}, \overline{q}_{ij}\right\}$ *for which* (5) *holds, and for any given* Π_{ij}, $(\overline{ic}_{ij})$ *and* $(\underline{ic}_{ij})$ *are satisfied.*

Conditional on type ij being correctly reported at the first stage, (5) is necessary for truthtelling to be induced at the second stage as well.

The next result, which is also rather obvious, evidences the circumstance under which, for each type, production levels are differentiated between states of nature, which means that P does not need to introduce inflexible rules to make the contract incentive-compatible.

Lemma 2 *At the solution to* Γ, $\underline{q}_{ij} > \overline{q}_{ij}$ $\forall ij \in \Upsilon$ *if and only if* σ_j *is sufficiently large.*

In the light of Lemma 2, we introduce the following assumption to ensure that, whatever his type, the agent is not imposed a standard rule at the second stage, and can thus exert discretion in the choice of a specific contractual option within the menu offered by P at the first stage.

Assumption 1 *For all* $j \in \{L, H\}$, σ_j *is sufficiently large in the sense of Lemma* 2.

Let us next focus on first-stage incentives. To understand how P is to tackle them, it is useful to construct a ranking that reflects how efficient each type is in production relative to the others. At this aim, we observe that the expected cost of production for the $ij-$agent reporting $i'j'$ is written as

$$EC_{ij}(q_{i'j'}, r_{i'j'}) = \theta_i q_{i'j'} - \sigma_j r_{i'j'}, \tag{6}$$

where

$$q_{i'j'} \equiv \frac{1}{2}(\underline{q}_{i'j'} + \overline{q}_{i'j'}) \ \text{ and } \ r_{i'j'} \equiv \frac{1}{2}(\underline{q}_{i'j'} - \overline{q}_{i'j'})$$

respectively denote the expected production level and the expected difference between the good and the bad-state production levels that P commends when she receives the report $i'j'$. Using the expression in (6), for any generic $(q, r)-$pair, we find that

$$EC_{LH}(q, r) < EC_{LL}(q, r) < EC_{HH}(q, r) < EC_{HL}(q, r). \tag{7}$$

According to (7), LH and HL are the most and the least efficient type, respectively. Types LL and HH both display an intermediate degree of efficiency, yet LL is more efficient than HH as $q > r$ (by definition) and $\Delta\theta > \Delta\sigma$ (by assumption).

3.2 Information rents and reduced problem

Hinging on (6), $IC_{ij}^{i'j'}$ can be reformulated as

$$\Pi_{ij} \geq \Pi_{i'j'} + EC_{ij}(q_{i'j'}, r_{i'j'}) - EC_{i'j'}(q_{i'j'}, r_{i'j'}),\ \forall ij, i'j' \in \Upsilon. \tag{8}$$

As in other multidimensional screening problems (Armstrong and Rochet [2], for instance), for all but the least efficient type, at least one downward incentive constraint is binding, "downward" here reflecting the temptation to pretend to be less efficient according to the ranking in (7) . This explains why, at optimum, the rents accruing to the four types are expressed as reported below.

Lemma 3 *At the solution to* Γ, *information rents are such that*

$$\Pi_{HL} = 0 \tag{9a}$$
$$\Pi_{HH} = \Delta\sigma r_{HL} \tag{9b}$$
$$\Pi_{LL} = \beta\Pi_{LL,1} + (1-\beta)\,\Pi_{LL,2} \tag{9c}$$
$$\Pi_{LH} = \gamma_1\Pi_{LH,1} + \gamma_2\left[\beta\Pi_{LH,2} + (1-\beta)\,\Pi_{LH,3}\right] + \gamma_3\Pi_{LH,4}, \tag{9d}$$

where

$$\Pi_{LL,1} = \Delta\theta q_{HL} \quad \textit{and} \quad \Pi_{LL,2} = \Delta\theta q_{HH} + \Delta\sigma r_{HL} - \Delta\sigma r_{HH} \tag{10}$$

together with

$$\begin{aligned} \Pi_{LH,1} &= \Delta\theta q_{HH} + \Delta\sigma r_{HL}, \quad \Pi_{LH,2} = \Delta\theta q_{HL} + \Delta\sigma r_{LL}, \\ \Pi_{LH,3} &= \Delta\theta q_{HH} + \Delta\sigma r_{HL} - \Delta\sigma r_{HH} + \Delta\sigma r_{LL} \quad \textit{and} \quad \Pi_{LH,4} = \Delta\theta q_{HL} + \Delta\sigma r_{HL}, \end{aligned} \tag{11}$$

and $\beta \in [0,1]\,,\ \gamma_z \in [0,1]\ \forall z \in \{1,2,3\}\,,\ \gamma_1 + \gamma_2 + \gamma_3 = 1.$

Having the rents as described in Lemma 3 does not necessarily mean that no upward incentive constraint is binding, and that production levels are not pooled, at optimum, for some types. However, when this does occur, the solution to Γ is, as usual, analogous to that to the following reduced problem:

$$\underset{\left\{\underline{q}_{ij},\,\overline{q}_{ij};\Pi_{ij}\right\}}{Max} \sum_{ij\in\Upsilon}\left\{\frac{1}{2}E_{ij}\left[\left(S(\underline{q}_{ij}) - (\theta_i - \sigma_j)\,\underline{q}_{ij}\right) + \left(S(\overline{q}_{ij}) - (\theta_i + \sigma_j)\,\overline{q}_{ij}\right)\right] - E_{ij}\left[\Pi_{ij}\right]\right\}. \tag{Γ'}$$

Lemma 4 *Suppose that the solution to* Γ' *solves* Γ *as well. Then, type* LH *is required to* *produce* $\underline{q}^*_{LH}$ *and* $\overline{q}^*_{LH}$, *whereas the production levels assigned to types* LL, HH *and* HL *satisfy*

$$\begin{aligned} S'(\underline{q}_{LL}) &= \theta_L - \sigma_L + \gamma_2 \frac{1-\mu}{\mu} \Delta\sigma \\ S'(\overline{q}_{LL}) &= \theta_L + \sigma_L - \gamma_2 \frac{1-\mu}{\mu} \Delta\sigma, \end{aligned}$$

$$\begin{aligned} S'(\underline{q}_{HL}) &= \theta_H - \sigma_L + \frac{\nu}{1-\nu} \left\{ \left[\beta + (\gamma_2\beta + \gamma_3) \frac{1-\mu}{\mu} \right] \Delta\theta + \left[1 - \beta + (1 - \nu\gamma_2\beta) \frac{1-\mu}{\nu\mu} \right] \Delta\sigma \right\} \\ S'(\overline{q}_{HL}) &= \theta_H + \sigma_L + \frac{\nu}{1-\nu} \left\{ \left[\beta + (\gamma_2\beta + \gamma_3) \frac{1-\mu}{\mu} \right] \Delta\theta - \left[1 - \beta + (1 - \nu\gamma_2\beta) \frac{1-\mu}{\nu\mu} \right] \Delta\sigma \right\} \end{aligned}$$

and

$$\begin{aligned} S'(\underline{q}_{HH}) &= \theta_H - \sigma_H + \frac{\nu}{1-\nu} \left\{ \left[\gamma_1 + (1-\beta) \left(\gamma_2 + \frac{\mu}{1-\mu} \right) \right] \Delta\theta - (1-\beta) \left(\gamma_2 + \frac{\mu}{1-\mu} \right) \Delta\sigma \right\} \\ S'(\overline{q}_{HH}) &= \theta_H + \sigma_H + \frac{\nu}{1-\nu} \left\{ \left[\gamma_1 + (1-\beta) \left(\gamma_2 + \frac{\mu}{1-\mu} \right) \right] \Delta\theta + (1-\beta) \left(\gamma_2 + \frac{\mu}{1-\mu} \right) \Delta\sigma \right\}, \end{aligned}$$

respectively.

As expected, quantities are (possibly) distorted away from the FB levels for all but the most efficient type (LH) in both states of nature. Based on this result, we can further investigate whether and under which circumstances the solution to Γ' solves Γ as well.

3.3 What matters in the determination of the solution

From Lemma 3 it is clear that Γ displays similarities with multidimensional screening problems, such as the one that Armstrong and Rochet [2] analyze. Albeit the way in which those problems are solved is somewhat suggestive of the procedure to follow in the framework here considered, identifying the set of binding incentive constraints is, yet, far from obvious. As compared to the problems aforementioned, additional complications ensue here from the interdependence between the values of q_{ij} and r_{ij} that enter the rents. This makes rather particular both the way in which the solution is found and the elements that matter at determining it. In essence, the solution reflects finely (i) the relative importance of the two knowledge imperfections (about expected cost and cost variability) that P suffers from, and (ii) the characteristics of the principal's surplus function. As it will become apparent in a while, depending upon how these elements combine, it might even happen that, at optimum, unlike in "standard" frameworks, some upward incentive constraint is binding, or production levels are bunched for some types.

The relative importance of the two knowledge imperfections is expressed by the ratio $\Delta\theta/\Delta\sigma$, which we denominate "spread index," and does not come as a surprise *per se*. Peculiar is here the way in which this contributes to shaping the optimal mechanism. That is, any change in the quantity that type ij is assigned in the bad state ($\overline{q}_{ij}$) triggers opposite effects in the two rent components $\Delta\theta q_{ij}$ and $\Delta\sigma r_{ij}$.

The characteristics of the surplus function, and, more precisely, the shape of S', are relevant precisely due to the problem of P being both sequential and multidimensional. In short, the shape of S' matters, first, because quantities are fixed considering the marginal surplus in expected terms at the first stage, which explains the relevance of sequentiality; second, because quantities are to be properly ranked not only between states of nature (for each type) but also across types (in each state), which explains the relevance of multidimensionality. To clarify this point, consider the FB scenario and observe that q^*_{ij} and r^*_{ij} are not necessarily ordered according to the types' efficiency ranking, as it is usually the case in screening problems (and, in particular, in problems that are either purely sequential or purely multidimensional). More precisely, FB expected productions (q^*_{ij}) are not necessarily ordered according to the efficiency ranking in (7), in spite of the expected cost (as defined in (6)) increasing with q. FB expected differences (r^*_{ij}) are not necessarily ordered inversely to (7), in spite of the expected cost decreasing with r. Indeed, the FB allocation is such that

$$\frac{1}{2}(S'(\underline{q}^*_{ij}) + S'(\overline{q}^*_{ij})) = \theta_i \tag{12}$$

$$\frac{1}{2}(S'(\overline{q}^*_{ij}) - S'(\underline{q}^*_{ij})) = \sigma_j, \tag{13}$$

from which we deduce that $q^*_{LH} \geq q^*_{HL}$ if and only if S' is convex, whereas $r^*_{HH} \geq r^*_{LH}$ if and only if S' is concave. Under asymmetric information, for truthtelling to be induced, the ranking of q_{ij} must respect the ranking in (7) and that of r_{ij} the converse order. This clearly means that how costly it is to elicit information, it depends upon the curvature of S'.

3.4 When the reduced problem solves (and when it does not) the general problem

The joint relevance of the spread index and of the features of the surplus function will become already evident in the two results that we state hereafter. They identify the conditions, in terms of magnitude of $\Delta\theta/\Delta\sigma$ and curvature of S', under which the list of quantities characterized in Lemma 4 as the solution to Γ', solves the general problem Γ as well, in contrast with those under which this is not the case.

Lemma 5 *At the solution to Γ, type HH and type HL are assigned the production levels in Lemma 4 only if the spread index is "small" i.e.,*

$$\frac{\Delta\theta}{\Delta\sigma} < \frac{\frac{1}{\nu} + (\mu - \beta)\left(\gamma_2 + \frac{\mu}{1-\mu}\right)}{\left|1 - \gamma_1 - (1-\beta)\left(\gamma_2 + \frac{\mu}{1-\mu}\right)\right|}. \tag{14}$$

Then, $\underline{q}_{HH} > \underline{q}_{HL}$ and $\overline{q}_{HL} > \overline{q}_{HH}$. Otherwise, $\underline{q}_{HL} = \underline{q}_{HH}$ and $\overline{q}_{HL} = \overline{q}_{HH}$.

The solution in Lemma 4 applies to Γ as long as (14) is satisfied. When (14) is not, the production level of type HL is pooled with that of type HH in either state. This follows from the difficulty that P faces in eliciting information from type LL. To see this, suppose that the rents are such that $\beta = 1$ at optimum (*i.e.*, $\Pi_{LL,1} > \Pi_{LL,2}$). Then, P is concerned with containing the rent that type LL grasps for not announcing HL (rather than HH), and needs to consider that the quantity chosen for type HL in the bad state affects the rents designed for types LL and HH in opposite directions. As $\overline{q}_{HL}$ is reduced, the former is contained, the latter is raised. Of course, the convenience for P to reduce $\overline{q}_{HL}$ depends upon the relative strength of these two effects, which is captured by the magnitude of the spread index. When the index is large, meaning that the agency cost associated with the possibility of type LL overstating θ is important, relative to that associated with the possibility of type HH understating σ, P would like to decrease $\overline{q}_{HL}$ below $\overline{q}_{HH}$, in addition to fixing $\underline{q}_{HH} < \underline{q}_{HL}$. However, the rent conceded for the report HH being relatively high, type LL would then display an incentive to overstate both θ and σ. This contradiction can be viewed formally by observing that $\Pi_{LL,1} > \Pi_{LL,2}$ is equivalent to

$$\Delta\sigma(r_{HH} - r_{HL}) > \Delta\theta(q_{HH} - q_{HL}), \tag{15}$$

hence to

$$(\Delta\theta + \Delta\sigma)(\overline{q}_{HL} - \overline{q}_{HH}) > (\Delta\theta - \Delta\sigma)(\underline{q}_{HH} - \underline{q}_{HL}), \tag{16}$$

and that (16) is violated if $\overline{q}_{HL} < \overline{q}_{HH}$ jointly with $\underline{q}_{HH} < \underline{q}_{HL}$. A similar contradiction is found by supposing that, on the opposite, $\beta = 0$ at optimum (*i.e.*, $\Pi_{LL,2} > \Pi_{LL,1}$) so that P is concerned with containing the rent that type LL grasps for not announcing HH (rather than HL). Then, with $\Delta\theta/\Delta\sigma$ large, she would like to decrease $\underline{q}_{HH}$ below $\underline{q}_{HL}$, which would yet trigger the temptation of type LL to exaggerate θ. This explains why, when $\Delta\theta/\Delta\sigma$ is so big to violate (14), the best for P is to force types HL and HH to produce equal quantities in both states.

Prior to stating the next result, it is useful to define the value of the spread index for which $\overline{q}_{LH} = \overline{q}_{HL}$ in Lemma 4 as

$$\delta \equiv \frac{(1 - \nu\gamma_2)(1 - \mu) + \mu(1 - \nu)}{\mu + \nu(1 - \gamma_1)(1 - \mu)}, \tag{17}$$

and to notice that, at optimum,

$$\delta < \frac{\frac{1}{\nu} + (\mu - \beta)\left(\gamma_2 + \frac{\mu}{1-\mu}\right)}{\left|1 - \gamma_1 - (1 - \beta)\left(\gamma_2 + \frac{\mu}{1-\mu}\right)\right|}. \tag{18}$$

To see why, consider that, whatever the size of the spread index, at the solution to Γ, type LH is required to produce more than type HH in the bad state $(\overline{q}_{LH} > \overline{q}_{HH})$. As long as (14) is satisfied so that $\overline{q}_{HL} > \overline{q}_{HH}$, type LH can be required to produce either more or less than type HL in the bad state. By contrast, when (14) is violated so that $\overline{q}_{HH} \geq \overline{q}_{HL}$, type LH is clearly assigned more output than type HL in the bad state. For production levels to be ranked in this way at the solution, it must be the case that (18) holds, indeed.

Lemma 6 *At the solution to* Γ, *types* LH *and* HL *are assigned the production levels in Lemma 4 only if* S' *is not very concave or, if it is,* $\frac{\Delta\theta}{\Delta\sigma} \neq \delta$. *Then,* $r_{LH} > r_{HL}$ *and* $q_{LH} > q_{HL}$. *When* S' *is very concave, if* (14) *holds, then* $q_{LH} = q_{HL}$ *for* $\frac{\Delta\theta}{\Delta\sigma} < \delta$ *and* $r_{LH} = r_{HL}$ *for* $\frac{\Delta\theta}{\Delta\sigma} > \delta$; *if* (14) *does not hold, then* $r_{LH} = r_{HL}$.

Having S' very concave means that P prefers not to induce much quantity dispersion, and is especially concerned with keeping production levels close across types in the good state. Under these circumstances, a particularly interesting situation arises in that an upward incentive constraint is binding, at optimum, either for type HH or for type LL, which might be tempted to declare LH. First suppose that the spread index satisfies (14) and is below δ. Then, if the solution in Lemma 4 were to apply, quantities would be such that $\overline{q}_{HL} > \overline{q}_{LH}$, the pronounced concavity of S' reflecting the desire of P to set quantities such that $\overline{q}_{HL}$ is raised above $\overline{q}_{LH}$ more than $\underline{q}_{HL}$ lowered below $\underline{q}_{LH}$. Yet, in that case, the contract would not be incentive-compatible as type HH would grasp a bonus of $\Delta\theta\,(q_{HL} - q_{LH})$ by understating θ and so pretending to be more efficient. To avoid this, P adjusts the production levels of types LH and HL such that $q_{LH} = q_{HL}$. Next suppose that the spread index satisfies (14) and is above δ. Then, if the solution in Lemma 4 were to apply, quantities would be such that $\overline{q}_{LH} > \overline{q}_{HL}$, and P would be concerned with removing the incentive of type LL to declare LH. This is made by choosing production levels for types LH and HL such that $r_{LH} = r_{HL}$. Obviously, no issue arises when $\Delta\theta/\Delta\sigma$ exactly equals δ and so $\overline{q}_{HL} = \overline{q}_{LH}$. Not surprisingly, as the spread index becomes so large to violate (14), the only upward incentive constraint that remains relevant is the one whereby type LL not be tempted to overstate σ, and P still optimally fixes $r_{LH} = r_{HL}$.

4 Characterization of the optimal mechanism

We begin by identifying a first core feature of the optimal contract *i.e.*, how the type-LL rent is set depending upon the curvature of S'. Taken together with Lemma 5 and 6, this result will then enable us to classify the solutions that Γ attains in the various cases to be listed in the sequel of the analysis.

Lemma 7 *Suppose that* (14) *holds at the solution to* Γ. *Then,* $\beta < 1$ *if and only if* S' *is "sufficiently" convex.*

We know from Lemma 5 that $\Pi_{LL,1}$ and $\Pi_{LL,2}$ are not necessarily the same when (14) holds. To identify the rent that is actually assigned to type LL, we need to consider the shape of S' as well. To see why, recall that having S' little convex (or concave) means that P prefers to keep quantities close across types, especially (though not only) in the good state. Therefore, as S' becomes less convex/more concave, the wedge between the HH- and the HL-production level is optimally decreased in either state, but at a higher rate in the good state, leading to $\Pi_{LL,1} > \Pi_{LL,2}$. This is why the relevant rent of type LL is $\Pi_{LL,1}$ as long as S' is concave or little convex, but not otherwise. A numerical illustration of this result is provided here below.

Example 1 *Consider the surplus function* $S(q) = aq - q^{b+1}/(b+1)$, $a, b > 0$. $S'(q)$ *is convex for* $b < 1$, *linear for* $b = 1$. *When* S' *is almost linear* $(b \approx 1)$, *according to Lemma* 7, *if* (14) *is satisfied, then* $\beta = 1$ *so that* $\Pi_{LL,2} > \Pi_{LL,1}$. *Furthermore, in this case, according to Solution* 5 *below,* $\gamma_1 > 0$ *together with* $\gamma_2 > 0$ *i.e.,* $\Pi_{LH,1} = \Pi_{LH,2}$. *When* b *is strictly below* 1, *the degree of convexity of* S' *may be "sufficiently" high in the sense of Lemma 7 for a different solution to arise. Table* 1 *below summarizes numerical results for the case in which* $a = 15$, $\nu = \mu = 0.3$, $\theta_L = 4$, $\theta_H = 5$, $\sigma_L = 3$, $\sigma_H = 3.3$. *Condition* (14) *holds for both values of* b *considered. Instead,* (15) *holds for* $b = 1$ *but not for* $b = 0.5$, *in which case it can be said that* S' *is "sufficiently" convex in the sense of Lemma* 7.

b	γ_1	γ_2	*Condition* (14)	$\Delta\sigma(r_{HH} - r_{HL})$	$\Delta\theta(q_{HH} - q_{HL})$
1	0.826	0.174	$3.33 < 16.73$	0.75	0.5
0.5	0.948	0.052	$3.33 < 57.63$	15	16.5

Table 1: Numerical results for S' linear and S' sufficiently convex

As equipped with the results in Lemma 4 to 7, we are now ready to show how the solution to Γ specifies according to whether the spread index satisfies or violates (14), and to whether S' is "sufficiently" convex (in the sense of Lemma 7) or displays a different curvature. In presenting the solutions that arise in the various cases, to avoid redundancy, we report the optimal quantities only when the conditions under which Lemma 4 applies do not hold so that the quantity solution is different.

To help the reader fix ideas, we provide a road map of Case 1 to 3 below in Figure 1 to 3, where we synthesize the temptations that, at the solution to Γ, are relevant, respectively, for type LH, LL and HH as the marginal surplus function proceeds from very concave to sufficiently convex (in horizontal) and the spread index increases up to the value identified in (14) (in vertical). This, of course, says which incentive constraints are binding.

Case 1: $\Delta\theta/\Delta\sigma$ small and S' concave (not close to linear)

We begin by considering the most complex situation, in which, as described in Lemma 6, S' is very concave and types LH and HL are not assigned the production levels pinned down in the reduced problem.

Solution 1 *Suppose that* (14) *holds at the solution to* Γ, *and that* S' *is "very" concave. Then,* $\beta = 1$ *and*

(i) *for* $\frac{\Delta\theta}{\Delta\sigma} \leq \frac{1-\mu}{\mu+\nu(1-\mu)}$: $\gamma_1 = 0$, γ_3 *such that* $r_{LL} = r_{HL}$, *and* $\gamma_2 = 1 - \gamma_3$;

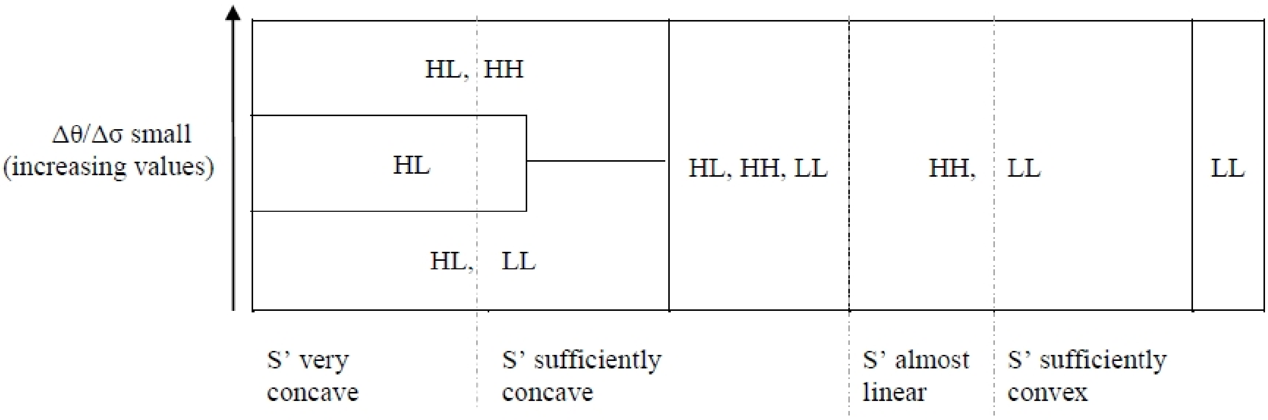

Figure 1: Relevant temptations for type LH in Case 1 to 3

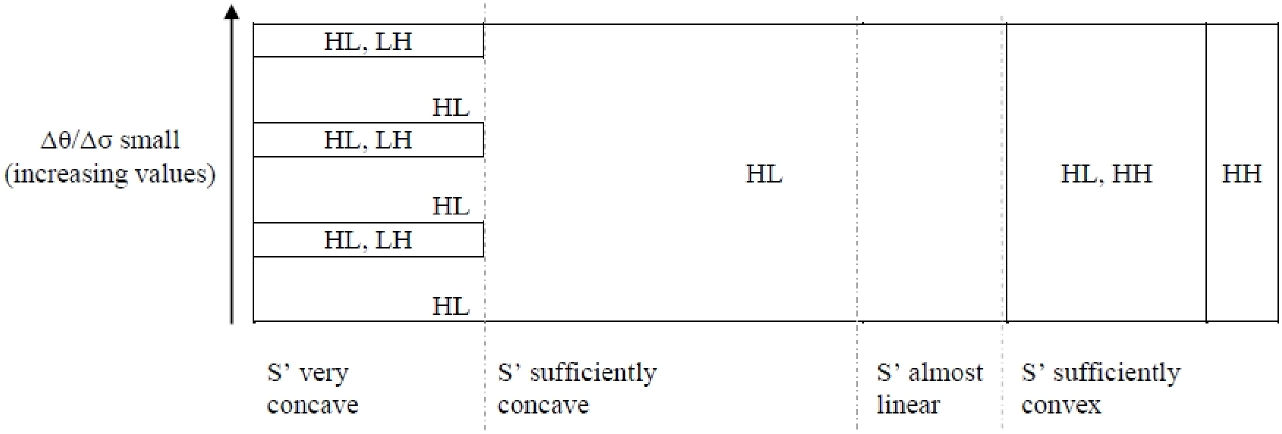

Figure 2: Relevant temptations for type LL in Case 1 to 3

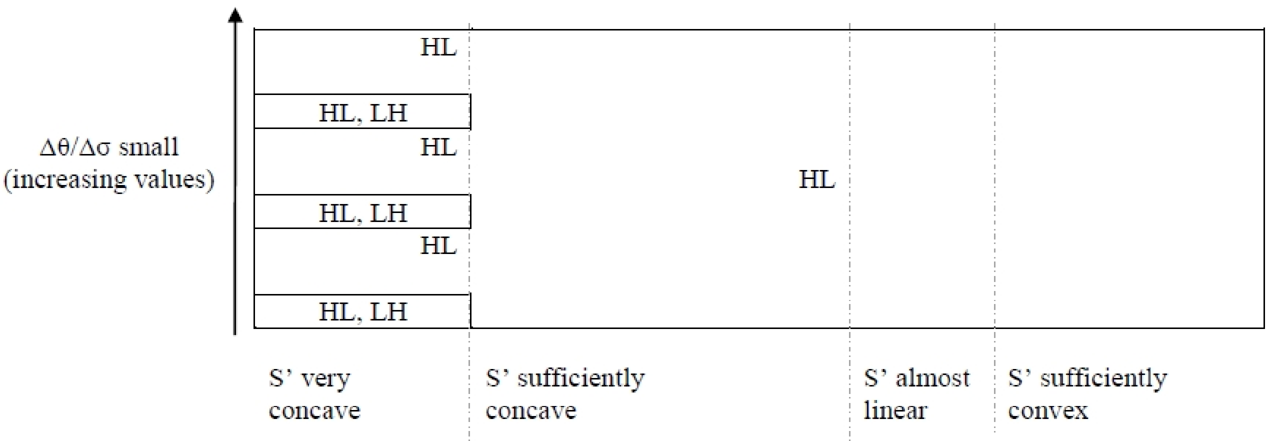

Figure 3: Relevant temptations for type HH in Case 1 to 3

(ii) *for* $\frac{1-\mu}{\mu+\nu(1-\mu)} < \frac{\Delta\theta}{\Delta\sigma} < \frac{1-\nu\mu}{\nu}$: $\gamma_3 = 1$;

(iii) *for* $\frac{\Delta\theta}{\Delta\sigma} \geq \frac{1-\nu\mu}{\nu}$: $\gamma_2 = 0$, γ_3 *such that* $q_{HH} = q_{HL}$, *and* $\gamma_1 = 1 - \gamma_3$.

In each of these scenarios, the production levels assigned to types HL *and* LH *satisfy*

$$S'(\underline{q}_{HL}) - \theta_H + \sigma_L - \lambda_1 = \frac{\nu}{1-\nu}\left(1 + (1-\gamma_1)\frac{1-\mu}{\mu}\right)\Delta\theta + \frac{(1-\gamma_2\nu)(1-\mu)}{\mu(1-\nu)}\Delta\sigma \quad (19)$$

$$S'(\overline{q}_{HL}) - \theta_H - \sigma_L - \lambda_1 = \frac{\nu}{1-\nu}\left(1 + (1-\gamma_1)\frac{1-\mu}{\mu}\right)\Delta\theta - \frac{(1-\beta\gamma_2\nu)(1-\mu)}{\mu(1-\nu)}\Delta\sigma \quad (20)$$

$$S'(\underline{q}_{LH}) - \theta_L + \sigma_H + \lambda_1 = 0 \quad (21)$$

$$S'(\overline{q}_{LH}) - \theta_L - \sigma_H + \lambda_1 = 0, \quad (22)$$

with λ_1 *such that* $q_{LH} = q_{HL}$, *if* $\frac{\Delta\theta}{\Delta\sigma} < \delta$, *and*

$$S'(\underline{q}_{HL}) - \theta_H + \sigma_L - \lambda_2 = \frac{\nu}{1-\nu}\left(1 + (1-\gamma_1)\frac{1-\mu}{\mu}\right)\Delta\theta + \frac{(1-\gamma_2\nu)(1-\mu)}{\mu(1-\nu)}\Delta\sigma \quad (23)$$

$$S'(\overline{q}_{HL}) - \theta_H - \sigma_L + \lambda_2 = \frac{\nu}{1-\nu}\left(1 + (1-\gamma_1)\frac{1-\mu}{\mu}\right)\Delta\theta - \frac{(1-\beta\gamma_2\nu)(1-\mu)}{\mu(1-\nu)}\Delta\sigma \quad (24)$$

$$S'(\underline{q}_{LH}) - \theta_L + \sigma_H + \lambda_2 = 0 \quad (25)$$

$$S'(\overline{q}_{LH}) - \theta_L - \sigma_H - \lambda_2 = 0, \quad (26)$$

with λ_2 *such that* $r_{LH} = r_{HL}$, *if* $\frac{\Delta\theta}{\Delta\sigma} > \delta$.

We learnt from Lemma 6 that, when S' is very concave, the quantity solution to Γ is not that in Lemma 4. Depending upon how large the spread index is relative to δ (and except for $\frac{\Delta\theta}{\Delta\sigma} \neq \delta$), one has either $q_{LH} = q_{HL}$ or $r_{LH} = r_{HL}$. The need to satisfy this additional constraint induces further distortions in the production levels assigned to types HL and LH, which are now characterized by (19) - (22), in the former case, and by (23) - (26), in the latter. Besides, Solution 1 has implications in terms of the exact rents that P assigns. The three scenarios it includes differ according to the incentives to cheat of type LH, which receives $\Pi_{LH,2}$ in (i), $\Pi_{LH,4}$ in (ii), $\Pi_{LH,1}$ in (iii). In scenarios (i) and (iii) the accruing rent equals $\Pi_{LH,4}$, whereas in scenario (ii) the rent is strictly larger than the potential benefit from any available lie. This means that P seeks to contain the rent that type LH grasps for not announcing HL or LL in (i), HL in (ii), HL or HH in (iii). To see why, recall that a decrease in $\overline{q}_{HL}$ has opposite effects in the rents that P concedes for not being announced LL or HH untruthfully, and that which effect dominates depends upon the magnitude of the spread index. Recall as well that, on efficiency grounds, a pronounced concavity of S' expresses the preference of P for keeping quantities close across types, and more in the good than in the bad state. For $\Delta\theta/\Delta\sigma$ very low, it is unnecessary to decrease $\overline{q}_{HL}$ largely below $\overline{q}_{LL}$, relative to how much $\underline{q}_{HL}$ is set below $\underline{q}_{LL}$, in order to contain the rent for not being reported LL (rather than HL), as type LH would gain little from overstating θ (in addition to understating σ). The best for P is to set the HL- and the LL-quantities close enough to have $r_{HL} = r_{LL}$, meaning that preventing type

LH from announcing HL is just as costly as preventing it from announcing LL. Similarly, for $\Delta\theta/\Delta\sigma$ relatively high, it is unnecessary to raise $\overline{q}_{HL}$ largely above $\overline{q}_{HH}$, relative to how much $\underline{q}_{HL}$ is fixed below $\underline{q}_{HH}$, in order to contain the rent for not being reported HH (rather than HL), as type LH would gain little from understating σ (in addition to overstating θ). The best for P is to fix the HL- and the HH-quantities close enough to have $q_{HL} = q_{HH}$, meaning that discouraging type LH from declaring HL is just as costly as discouraging it from declaring HH. When the spread index takes values as in (ii), overstating θ is nearly as attractive as understating σ, and HL remains the sole relevant temptation for type LH.

For the same range of values of the spread index, the solution becomes "more standard" when S' is not very concave because, then, there is no longer any type willing to mimic some more efficient type, and the quantities in Lemma 4 are such that $q_{LH} > q_{HL}$ and $r_{LH} > r_{HL}$. For type LH, initially, relevant incentive constraint is that whereby it not be tempted to declare HL, leading to the solution here below.

Solution 2 *Suppose that* (14) *holds at the solution to* Γ, *and that* S' *is "sufficiently" (though not very) concave. Then, Solution* 1 *still applies, except that types* LH *and* HL *are assigned the production levels reported in Lemma* 4 $(\lambda_1 = \lambda_2 = 0)$.

Then, as the concavity of S' gets progressively less pronounced, and the differences $q_{HL} - q_{HH}$ and $r_{HL} - r_{LL}$ reduce (as explained), the other incentive constraints of type LH, whereby it not be attracted by the contractual offers designed for types HH and LL, become relevant as well. This leads to the next two solutions.

Solution 3 *There exist some degrees of concavity of* S', *less pronounced than required for Solution* 2 *to arise, and some* $\delta_1 \in \left(\frac{1-\mu}{\mu+\nu(1-\mu)}; \frac{1-\nu\mu}{\nu}\right)$ *such that, at the solution to* Γ, $\beta = 1$ *and* $\gamma_3 > 0$ *together with either* $\gamma_1 > 0$ *and* $\gamma_2 = 0$ *(if* $\frac{\Delta\theta}{\Delta\sigma} > \delta_1$*) or* $\gamma_1 = 0$ *and* $\gamma_2 > 0$ *(if* $\frac{\Delta\theta}{\Delta\sigma} < \delta_1$*).* γ_3, γ_1 *and* γ_2 *are pinned down as in Solution* 1.

Solution 4 *Suppose that* S' *is less concave than required for Solution* 3 *to arise, though not close to linear. Then, at the solution to* Γ, $\beta = 1$, γ_2 *and* γ_3 *such that* $q_{HL} = q_{HH}$ *and* $r_{HL} = r_{LL}$, *and* $\gamma_1 = 1 - \gamma_2 - \gamma_3$.

Case 2: $\Delta\theta/\Delta\sigma$ small and S' (almost) linear

Case 2 includes one sole solution.

Solution 5 *Suppose that* (14) *and* (15) *hold at the solution to* Γ, *and that* S' *is either almost linear or linear. Then,* $\beta = 1$, γ_1 *such that*

$$\Delta\theta(q_{HH} - q_{HL}) = \Delta\sigma(r_{LL} - r_{HL}), \tag{27}$$

$\gamma_2 = 1 - \gamma_1$ *and* $\gamma_3 = 0$.

Type LH is assigned the same rent for not declaring HH and LL (*i.e.*, as from (27), $\Pi_{LH,1} = \Pi_{LH,2}$). By contrast, HL is no longer an attractive lie for this type. The reason is that, when S' is (almost) linear, P does not care about keeping q_{HL} larger than q_{HH} and r_{HL} larger than r_{LL}. The temptation to report HL is thus removed by fixing production levels such that $q_{HH} > q_{HL}$ and $r_{LL} > r_{HL}$. Nevertheless, P is unwilling to set q_{HH} and r_{LL} widely above q_{HL} and r_{HL}, respectively. Actually, (quasi-)linearity of S' means that P prefers the spread between production levels of types HL and HH and of types HL and LL to be not too different in the two states. Therefore, efficiency is optimally traded-off against rent-extraction purposes by choosing quantities such that, for type LH, announcing LL is exactly as appealing as announcing HH.

As S' becomes more convex, one might conjecture that either of the differences $q_{HH} - q_{HL}$ and $r_{LL} - r_{HL}$ increases faster than the other, hence that (27) is no longer satisfied. On the other hand, the raise in $q_{HH} - q_{HL}$ involves that (16) tightens, in turn. One thus wonders whether a solution can arise at which (27) is violated whereas (16) holds. The next lemma clarifies this point.

Lemma 8 *For any given degree of convexity of* S', *when the spread index is close to* 1, *at the solution to* $\Gamma : \gamma_1 > 0, \gamma_2 > 0, \gamma_3 = 0$ *and* $\beta = 1$. *For higher values of the spread index (though still such that* (14) *holds),* γ_1 *is higher,* γ_2 *lower, and the condition (16) under which* $\beta = 1$ *tighter. There exists no solution at which either* $\gamma_1 = 1$ *and* $\beta = 1$.

We first provide a numerical illustration for this result, and then discuss it.

Example 2 *Consider the surplus function used in Example* 1, *and take the same values of the parameters with the following two exceptions: b is fixed equal to* 0.5; σ_H *varies (together with* $\Delta\theta/\Delta\sigma$) *as shown in Table* 2 *below. As from Example* 1, (15) *is violated for* $\sigma_H = 3.3$. *Instead, it is not for* $\sigma_H = 3.6$ *and* $\sigma_H = 3.9$ *i.e., when* $\Delta\theta/\Delta\sigma$ *takes a lower value.*

σ_H	$\Delta\theta/\Delta\sigma$	γ_1	γ_2	$\Delta\sigma(r_{HH} - r_{HL})$	$\Delta\theta(q_{HH} - q_{HL})$
3.9	1.11	0.47	0.53	122	58.5
3.6	1.66	0.675	0.325	57	41
3.3	3.33	0.948	0.052	15	16.5

Table 2: Numerical results for increasing values of the spread index

From Lemma 7, we learnt what happens to (15) as S' becomes more convex, for any given value of the spread index. That is, (15) initially tightens and, as soon as S' becomes "sufficiently" convex, it is no longer satisfied. Additionally, Lemma 8 tells us what happens to (15) as the spread index varies, given the curvature of S'. Specifically, (15) becomes more stringent as the spread index increases above 1. At the same time, type LH displays stronger incentives to exaggerate θ, relative to understating σ, meaning that γ_1 raises at optimum. Yet, according to Lemma 8, γ_1 cannot reach 1 unless (15) is violated. Because (15) tightens also as

S' becomes more convex, it must be the case that the solution that follows Solution 5 arises for S' "sufficiently" convex in the sense of Lemma 7.

Case 3: $\Delta\theta/\Delta\sigma$ small and S' "sufficiently" convex

When S' is sufficiently convex, we distinguish two situations. As one could expect on the basis of the previous analysis, as long as S' is not very convex, HL and HH are both relevant temptations for type LL; HH remains the sole relevant temptation when the convexity of S' becomes very pronounced. Again, this is explained by the increasingly stronger preference that P displays for containing quantity distortions, especially for more efficient types (hence, for type HH relative to type HL). The next two solutions thus arise.

Solution 6 *Suppose that* (14) *holds at the solution to* Γ, *and that* S' *is "sufficiently" but not very convex. Then,* $\beta > 0$ *and such that* $r_{HH} = r_{LL}$, *together with* $\gamma_1 > 0$ *and* $\gamma_2 > 0$ *as pinned down in Solution* 5, *and* $\gamma_3 = 0$.

Solution 7 *Suppose that* (14) *holds at the solution to* Γ, *and that* S' *is more convex than required for Solution* 6 *to arise. Then,* $\beta = 0$ *together with* $\gamma_2 = 1$.

Case 4: $\Delta\theta/\Delta\sigma$ large

To complete the analysis, we now turn to explore the case in which the spread index is sufficiently large to violate (14). From Lemma 5 we know that, in this case, type HL is optimally required to provide the same output as type HH in either state, production levels of these types being no longer characterized as in Lemma 4. Except for this remarkable aspect, the solutions to Γ that can be regrouped in Case 4 do not display very different features from those previously presented. In short, for S' very concave, some upward incentive constraint is binding, as in Solution 1 (Case 1). As S' moves from sufficiently concave to almost linear and then to sufficiently convex, relevant temptations of type LH evolve from HL (as in Case 1) to HH and LL (as in Case 2), and then to LL (as in Solution 7, Case 3). In presenting the solutions formally, we omit the characterization of the HL-quantities, which are pooled with those designed for type HH. We further omit the quantities assigned to types LL and LH, which are pinned down as in Lemma 4, with the sole exception of Solution 8 for type LH. Lastly, the value of β is neglected as, with $\underline{q}_{HL} = \underline{q}_{HH}$ and $\overline{q}_{HL} = \overline{q}_{HH}$, the rent accruing to type LL always amounts to $\Delta\theta q_{HL} = \Delta\theta q_{HH}$.

Solution 8 *Suppose that* (14) *is violated at the solution to* Γ, *and that* S' *is sufficiently concave. Then,* $\gamma_1 = 1$. *The production levels assigned to type* HH *satisfy*

$$S'(\underline{q}_{HH}) = \theta_H - \sigma_H + \frac{\nu}{(1-\nu)(1-\mu)}\Delta\theta + \frac{1}{1-\nu}\Delta\sigma + \lambda_3 \qquad (28)$$

$$S'(\overline{q}_{HH}) = \theta_H + \sigma_H + \frac{\nu}{(1-\nu)(1-\mu)}\Delta\theta - \frac{1}{1-\nu}\Delta\sigma - \lambda_3 \qquad (29)$$

those assigned to type LH

$$
\begin{aligned}
S'(\underline{q}_{LH}) &= \theta_L - \sigma_H - \lambda_3 && (30)\\
S'(\overline{q}_{LH}) &= \theta_L - \sigma_H + \lambda_3, && (31)
\end{aligned}
$$

with $\lambda_3 > 0$ *and such that* $r_{HH} = r_{LH}$ *if* S' *is very concave, and* $\lambda_3 = 0$ *otherwise.*

Solution 9 *Suppose that* (14) *is violated at the solution to* Γ*, and that* S' *is (almost) linear. Then,* γ_2 *is such that* $r_{LL} = r_{HH}$*,* $\gamma_1 = 1 - \gamma_2$ *and* $\gamma_3 = 0$*. The production levels assigned to type HH satisfy*

$$
\begin{aligned}
S'(\underline{q}_{HH}) &= \theta_H - \sigma_H + \frac{\nu}{(1-\nu)(1-\mu)}\Delta\theta + \frac{1-\gamma_2\nu}{1-\nu}\Delta\sigma && (32)\\
S'(\overline{q}_{HH}) &= \theta_H + \sigma_H + \frac{\nu}{(1-\nu)(1-\mu)}\Delta\theta - \frac{1-\gamma_2\nu}{1-\nu}\Delta\sigma && (33)
\end{aligned}
$$

with γ_2 *such that* $r_{LL} = r_{HH}$.

Solution 10 *Suppose that* (14) *is violated at the solution to* Γ*, and that* S' *is sufficiently convex. Then,* $\gamma_2 = 1$*. The production levels assigned to type HH satisfy*

$$
\begin{aligned}
S'(\underline{q}_{HH}) &= \theta_H - \sigma_H + \frac{\nu}{(1-\nu)(1-\mu)}\Delta\theta + \Delta\sigma && (34)\\
S'(\overline{q}_{HH}) &= \theta_H + \sigma_H + \frac{\nu}{(1-\nu)(1-\mu)}\Delta\theta - \Delta\sigma. && (35)
\end{aligned}
$$

5 Results, interpretation and discussion

The analysis developed so far makes it clear that the characteristics of the marginal surplus function are key to determining the features of the optimal multidimensional-and-sequential screening mechanism. Not only the curvature of S' dictates the implications, in terms of efficiency, that are associated with the quantity decisions of P. It also says which incentives to lie P needs to be concerned with while choosing the allocation.

Proposition 1 *For any given value of the spread index, as* S' *becomes less concave/more convex, information rents and production levels in the optimal mechanism reflect increasingly stronger incentives to mimic less inefficient types:*

- *for type LL : HL in Case* 1 *and* 2*; HL together with HH and then HH in Case* 3*;*
- *for type LH :*
  - *with* (14) *satisfied: HL (possibly, together with HH and/or LL) in Case* 1*; HH and LL in Case* 2 *and Solution* 6*, Case* 3*; LL in Solution* 7*, Case* 3*;*
  - *with* (14) *violated: HL and HH in Solution* 8*; HL, HH and LL in Solution* 9*; LL in Solution* 10*.*

On the one hand, the shape of S' dictates how dispersed production levels should be on an efficiency ground. As we explained, little dispersion is desirable when S' is very concave and increasingly more dispersion as S' becomes less concave/more convex. On the other hand, from a rent-extraction perspective, P prefers to choose production levels relatively close both across types and between states of nature, because this makes first-stage cheating less appealing. Therefore, moving from concavity to convexity, the trade-off between rent-extraction and efficiency, the latter being expressed in expected terms over the two possible productions, is initially loose and then progressively exacerbated. In a unidimensional framework, whether sequential or not, only one temptation to cheat for each type (excluding the least efficient one) would matter in determining the allocation. One would not be faced with a multiplicity of relevant cases, and the various solutions would collapse onto a unique solution. The shape of S' would play no role from this perspective. In a multidimensional framework where screening did not occur sequentially, quantities would be determined at the first stage, one for each type. In that case, the efficiency losses associated with quantity distortions would not be evaluated in expected terms. Once again, the shape of S' would be irrelevant in the identification of the rents to be optimally assigned to the various types, and all cases/solutions to Γ that are differentiated according to the curvature of S' would collapse onto one single case/solution. By this we do not mean that, in multidimensional screening problems that are not sequential, the principal's preferences never affect contractual features. Sometimes this occurs, yet in different ways. In environments where the agent exerts two distinct activities for the principal and knows the cost of each activity privately, it is the preference symmetry across activities (rather than the shape of the marginal surplus function) that matters in the optimal contractual choice (Armstrong and Rochet [2]).

The result in Proposition 1 may sound slightly abstract. In fact, we can identify at least two simple interpretations for it.

To begin with, P can be viewed as the regulator of some industry, concerned with maximizing expected consumer surplus. Then, the marginal surplus function is the inverse demand function. For instance, in Example 1 and 2, the inverse demand function is $S'(q) \equiv p(q) = a - q^b$, concave in q if $b > 1$, linear if $b = 1$, convex if $b < 1$. It is immediate to check that, in this case, all else equal, the direct demand, namely $q(p) = (a-p)^{1/b}$, is less price-elastic the larger b, for any given p. This evidences one first relevant aspect *i.e.*, the move from concavity to linearity to convexity corresponds, in fact, to a shift from a more to a less price-elastic demand. When b is large, thus demand is little elastic, consumers are ready to pay more for not renouncing to some given amount of the product. Therefore, in principle, when faced with a little elastic demand, the regulator should design a policy that prevents consumption from diverging significantly from the FB quantity level because that would come with an important surplus loss. This is in line with the prediction of our model that, at optimum, P becomes less prone to distort production levels as S' switches from concave to convex and, to avoid that sacrifice, she tolerates more important agency costs (*i.e.*, she concedes rents for removing incentives to mimic less inefficient types). Moreover, the regulator being informed neither about the type of the agent/firm nor about the final state of nature, at the time when she designs the policy, she does not know yet the exact point, along the demand curve, at which the agent will ultimately operate. To see why this is important, consider that, for any given b in our example, consumers are ready to pay more for escaping further reduction when the quantity is small than they would if the initial quantity were big. This involves that letting consumption diverge from the FB output level when the latter is large wastes more surplus than it does when it is small. Remarkably, this "local" effect is more pronounced the larger b. Altogether, this suggests that, the regulator targeting consumer surplus in expected terms, the exact output schedule cannot be properly structured, especially when demand is little elastic, unless the surplus losses that would follow from distortions are traded-off against one another, depending upon the exact point where FB output levels lie along the demand curve. This comes back to the other prediction of our model that, at optimum, P distorts production levels less with a convex than with a concave marginal surplus function, particularly when she faces little inefficient types and the realized state is good, which are the circumstances under which more output should be produced.

The very interest of the relationship between the curvature of the marginal surplus function and the price-elasticity of market demand resides, perhaps, in that it evidences a way to make functional use of the insights of our study along the current regulatory practice. Actually, in markets where the demand function is vaguely known to regulators, the latter typically refer to elasticity estimates, which can be formed with more reasonable accuracy when little information is available about demand conditions. This suggests that, specifically in regulator/firm hierarchies where the information structure is as represented in our model, the regulator could use elasticity estimates to identify the relevant information rents and fix output accordingly.

Another way to read the result in Proposition 1 emerges once $S(q)$ is reinterpreted as a utility function. According to the theory of consumption under uncertainty, an individual whose marginal utility function (S' in our model) is convex with respect to consumption, is prudent and engages in precautionary saving. This means that she would strongly dislike to have very low consumption in the future, hence she saves more at present to avoid that this outcome is realized. By contrast, an individual whose marginal utility function is concave, takes a dissaving behaviour (see Leland [16] for a by-now classical contribution; for more recent work, see Melegatti [17], for instance). In our framework, the case of convex marginal surplus could be viewed as one in which the principal is prudent with respect to output provision, that of concave marginal surplus as one in which the principal is not, and the presence of uncertainty about the final cost realization as leading to a more or less precautionary attitude. When the marginal surplus function is convex, the principal is especially concerned with the output that she will be delivered at the second stage. Not only it should not be too little, whatever the realized state. Also, production in the good state (*i.e.*, less costly production) should be privileged with respect to production in the bad state (*i.e.*, more costly production). The principal gives up more surplus at the first stage, in return for higher future production, especially when a low cost is realized. When the marginal surplus function is concave, the principal is less concerned with output. She does not seek to ensure herself against the event that a very small quantity be ultimately provided, even in the good state. She can thus retain more surplus from the agent at the contracting stage.

Given the content of Proposition 1, and in the light of the explanations that we have been providing, it should by now be clear that the screening problem that P tackles shares no other notable similarity with the multidimensional screening problems previously explored than the information rents associated with binding downward incentive constraints reported in Lemma 3. We previously explained that the interdependence between q_{ij} and r_{ij}, which are the counterpart for, but do not have the same nature as, the production levels of two distinct activities in existing models, is the complication that causes the solution to be strictly related to the curvature of S' in our framework. We further pointed out that, as S' becomes less concave/more convex, P finds it progressively more costly to distort quantities in the good state relative to the bad state, involving that, for any given ij, r_{ij} should be raised, at optimum, relative to q_{ij}. Taking Proposition 1 and all these considerations into account, the following corollary can be stated.

Corollary 1 *For any given value of the spread index, as S' becomes less concave/more convex, the evolution of the optimal information rents reflects the temptation to overstate θ progressively lessening, relative to the temptation to understate σ.*

It is further noteworthy that, among the various solutions to Γ, some display a few common features with the sequential screening mechanism that would be optimal in a unidimensional framework. First, this occurs when $\Delta\theta/\Delta\sigma$ is large, which is not surprising in the light of Lemma 5, according to which P pools the production levels of the θ_H-types with respect to the second dimension of private information. A second similarity shows up for whatever size of the spread index, hence also for $\Delta\theta/\Delta\sigma$ small. This is perhaps less obvious, provided P does design fully flexible production rules (*i.e.*, differentiates quantities with respect to both θ and σ) when $\Delta\theta/\Delta\sigma$ is small. The similarity resides in that, in nearly all solutions, output is fixed at the FB level for some low-expected-cost-type, which is the counterpart for the no-distortion-at-the-top result in Riordan and Sappington [20] and Courty and Li [9]. In spite of these common features, as long as $\Delta\sigma > 0$, the optimal mechanism in our setting never reduces to the sequential screening mechanism that would be optimal if θ were the sole piece of private information.

Corollary 2 *As long as $\Delta\sigma > 0$, there is no value of the spread index and no curvature of S' for which the optimal mechanism collapses onto a unidimensional-and-sequential screening mechanism:*

- *as long as* (14) *holds, the optimal output profile is fully separating for all $ij \in \Upsilon$; otherwise, it is for types Lj, whereas pooling is induced for types Hj, $\forall j \in \{L, H\}$;*
- *in scenario* (i) *of Solution* 1, *quantities are distorted away from FB levels for both θ_L-types; in all other solutions, only one such type is assigned the FB production levels.*

6 *Ex-post* participation constraints

We developed the analysis focusing on the case in which P makes sure that the agent breaks even in expectation by choosing a policy that satisfies the first-stage participation constraint. A more realistic situation would be that in which P warrants that the agent breaks even in all possible states of nature. This would require to solve a new problem Γ_{ep}, in which the *ex-post* participation constraints

$$\begin{aligned} \underline{\pi}_{ij} &\geq 0 \quad (\underline{pc}_{ij}) \\ \overline{\pi}_{ij} &\geq 0 \quad (\overline{pc}_{ij}) \end{aligned}$$

replace (PC_{ij}) for all $ij \in \Upsilon$. In spite of the practical relevance, the introduction of these constraints brings about analytical complications without affecting the very structure of the optimal mechanism in any significant respect. This explains why we preferred to work with the first-stage participation constraint in the first place.

To evidence the similarity with the optimal mechanism under Γ in a parsimonious manner, we hereafter present the solution to Γ_{ep} with reference to one specific situation, that in which the spread index is small (according to (36) below), S' (almost) linear, and the sole binding *ex-post* participation constraint is that of type HL in the bad state (see Appendix G for mathematical details).

Solution 11 *Suppose that, at the solution to* Γ_{ep}, $(\overline{pc}_{ij})$ *is binding for* $ij = HL$ *and slack for all* $ij \neq HL$. *Further suppose that the spread index is "small" i.e.,*

$$\frac{\Delta\theta}{\Delta\sigma} < \frac{1}{1-\gamma_1}\left[\frac{1}{\nu} - \gamma_2(1-\mu) - \mu\left(1 + \frac{1-\nu}{\nu}\frac{\sigma_L}{\Delta\sigma}\right)\right], \tag{36}$$

and that S' *is (almost) linear. Then, the solution to* Γ_{ep} *is pinned down as in Lemma* 3 *and* 4, *with* $\beta = 1$, γ_1 *satisfying* (27), *and* $\gamma_2 = 1 - \gamma_1$, *except that*

$$\begin{aligned} \Pi_{HL} &= \sigma_L \overline{q}_{HL} \\ S'(\overline{q}_{HL}) &= \theta_H + 2\sigma_L + \frac{\nu}{1-\nu}\left[\left(1 + \gamma_2\frac{1-\mu}{\mu}\right)\Delta\theta - (1-\gamma_2\nu)\frac{1-\mu}{\nu\mu}\Delta\sigma\right]. \end{aligned}$$

Solution 11 is reminiscent of Solution 5, which was found to be optimal in Γ for $\Delta\theta/\Delta\sigma$ satisfying (14) and S' (almost) linear. The similarity rests on the circumstance that the same incentive constraints are binding at the two solutions.

The observations that are made and the results that are found in other sequential (but unidimensional) screening models cast doubts on the optimality of sequential screening when *ex-post* participation constraints are imposed. Courty and Li [9] acknowledge that, when the agent cannot be exposed to losses *ex post*, sequentiality yields no benefit and the principal should rather screen types *ex post* (footnote 8, p.706). In a more recent study, Krahmer and Strausz [15] prove this formally. They further identify the reason why sequential screening is beneficial, absent *ex-post* outside options, in that it relaxes participation (rather than incentive compatibility) constraints. Both in Courty and Li [9] and in Krahmer and Strausz [15], as essential aspect is that the agent's first-stage information is a signal about the final state *i.e.*, it affects the realization of the latter only stochastically. As a consequence, the number of possible states remains unchanged even after the first-stage information is revealed. By contrast, in our model, the agent's first-stage information affects the cost realization in a direct way: once type ij is identified, the final cost cannot take other values than $\theta_i - \sigma_j$ and $\theta_i + \sigma_j$. That is, conditional on the first-stage type being identified, the set of possible realizations (and so the set of incentive constraints to be satisfied) shrinks significantly. Under these circumstances, sequential screening is still preferable to *ex-post* screening. We hereafter provide a numerical example that illustrates the desirability of sequential screening with regards to the situation to which Solution 11 refers.

Example 3 *Take again the surplus function considered in Example* 1 *and* 2*. Let* $a = 15$ *and* $b = 1$ *(*S' *linear). Further take* $\nu = 0.3$, $\mu = 0.2$, $\theta_L = 4$, $\theta_H = 5.2$, $\sigma_L = 3$, $\sigma_H = 3.8$*. Then, the spread index equals* 1.5 *and satisfies* (36)*. At the solution to* Γ_{ep}, $\gamma_1 = 0.9$ *and* $\gamma_2 = 0.1$*. Table* 3 *below summarizes the results that are obtained in this environment under sequential and ex-post screening, showing that the former yields a higher payoff to P.*[12]

	Sequential screening				**Ex-post screening**			
	LH	LL	HH	HL	LH	LL	HH	HL
$\underline{q}$	14.8	13.7	13.13	7.65	14.8	13.8	13.48	12.1
$\overline{q}$	7.2	8.3	5.53	7.52	5.37	5.37	5.37	5.37
Π	33.8	31.6	22.6	22.57	34.5	31.17	23.1	20.4
W	36.16				30.81			

Table 3: Comparison between sequential and ex-post screening

One natural application of our model with *ex-post* participation constraints is the award of a contract for monopoly franchise, which is frequently made through an auction. Such constraints are relevant when the contract is signed in an economic context characterized by enforcement imperfections. Recent studies about contracts between governments and private firms for public service provision evidence how important non-enforceability is in the design of public-private relationships (see, in particular, Guasch [14]). However, the private information that the bidders hold at the auctioning stage is definitely not less problematic an aspect. Our work contributes to the literature on the award of monopoly franchises by considering the very realistic situation in which, at the bidding stage, auction participants and, in particular, the firm that ultimately wins the tender, know privately not only the expected value of the project but also the entire value distribution. For the sake of completeness, in Appendix G.1

[12] In this particular example, under *ex-post* screening, all types are imposed the same quantity (namely, 5.37) in the bad state. This is because the monotonicity condition $\overline{q}_{LL} \geq \overline{q}_{LH}$ would be violated otherwise. Further details are relegated to Appendix H.3.

we report the main features of the optimal mechanism in the auction framework for the case of S' (almost) linear. Recalling that the curvature of the marginal surplus function can be interpreted in terms of price-elasticity of demand, the focus on the (almost-)linear case is meant to capture the circumstance that, in markets for services of general interest, demand is not highly sensitive to price variations.

7 Conclusion

We showed how to characterize and interpret the optimal screening mechanism in a principal-agent relationship in which, at the contracting stage, the agent is privately informed about both the expected value and the variability of the production cost and, at a later stage, he learns the realized cost and produces some given good for the principal. This framework represents a variety of real-world situations, such as regulatory and procurement contexts, in which the cost of performing the concerned activity is uncertain when the contractual relationship begins and the activity is not executed until after uncertainty is solved.

With this work, we contribute to the study of sequential screening problems, which has recently gained significant momentum in the literature, precisely in that we consider the truly realistic case in which the agent holds private information on the whole set of possible values of the parameters that matter in the relationship with the principal, rather than only on an aggregate of those values. The predictions of the analysis evidence that, in this case, for information rents to be optimally structured and ordered across agent's types, it is necessary to carefully consider the preferences of the principal or, more generally, the preferences of the economic players on behalf of whom the principal acts. This, for instance, means that, prior to organizing auctions for monopoly franchises in frameworks of the kind that our model represents, governments and/or regulators should examine consumer preferences for the concerned goods meticulously, and base the screening task finely on the characteristics of those preferences. This is not beyond reach, in practice, as long as synthetic, yet instructive, indices (such as the elasticity to price) are available.

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A Preliminary analysis

A.1 Proof of Lemma 3

Using the definitions of q_{ij} and r_{ij}, (3) is rewritten as

$$\Pi_{ij} = \frac{1}{2}(\underline{t}_{ij} + \bar{t}_{ij}) - (\theta_i q_{ij} - \sigma_j r_{ij}),\ \forall ij \in \Upsilon.$$

Based on this expression, we reformulate the first-stage incentive constraints as follows:

$$\begin{aligned}
\Pi_{LL} &\geq \Pi_{HL} + \Delta\theta q_{HL} && \text{(IC1)}\\
\Pi_{LL} &\geq \Pi_{LH} - \Delta\sigma r_{LH} && \text{(IC2)}\\
\Pi_{LL} &\geq \Pi_{HH} + \Delta\theta q_{HH} - \Delta\sigma r_{HH} && \text{(IC3)}\\
\Pi_{HL} &\geq \Pi_{LL} - \Delta\theta q_{LL} && \text{(IC4)}\\
\Pi_{HL} &\geq \Pi_{HH} - \Delta\sigma r_{HH} && \text{(IC5)}\\
\Pi_{HL} &\geq \Pi_{LH} - \Delta\theta q_{LH} - \Delta\sigma r_{LH} && \text{(IC6)}\\
\Pi_{LH} &\geq \Pi_{HH} + \Delta\theta q_{HH} && \text{(IC7)}\\
\Pi_{LH} &\geq \Pi_{LL} + \Delta\sigma r_{LL} && \text{(IC8)}\\
\Pi_{LH} &\geq \Pi_{HL} + \Delta\theta q_{HL} + \Delta\sigma r_{HL} && \text{(IC9)}\\
\Pi_{HH} &\geq \Pi_{LH} - \Delta\theta q_{LH} && \text{(IC10)}\\
\Pi_{HH} &\geq \Pi_{HL} + \Delta\sigma r_{HL} && \text{(IC11)}\\
\Pi_{HH} &\geq \Pi_{LL} - \Delta\theta q_{LL} + \Delta\sigma r_{LL}. && \text{(IC12)}
\end{aligned}$$

Using the downward incentive constraints (IC1), (IC3), (IC7) to (IC9), and (IC11), and taking at least one downward incentive constraint to be binding for each type, except HL, together with PC_{HL}, the rents in (9a) to (9d) are obtained. The values of β and γ_z, $\forall z \in \{1,2,3\}$, are to be found at the solution to Γ. In each of the proofs of the various solutions below, it is shown that, with the rents in (9a) to (9d), $IC_{ij}^{i'j'}$ is satisfied for all $ij, i'j' \in \Upsilon$.

A.2 Proof of Lemma 4

Using the rents in Lemma 3, we rewrite the objective function of P as:

$$\begin{aligned}
&\sum_{ij\in\Upsilon} E_{ij}\left[\frac{1}{2}\left(S(\underline{q}_{ij}) + S(\overline{q}_{ij})\right) - (\theta_i q_{ij} - \sigma_j r_{ij})\right] \qquad (37)\\
&-\nu\mu\left[\beta\Delta\theta q_{HL} + (1-\beta)\left(\Delta\sigma r_{HL} + \Delta\theta q_{HH} - \Delta\sigma r_{HH}\right)\right]\\
&-(1-\nu)(1-\mu)\Delta\sigma r_{HL}\\
&-\nu(1-\mu)\left\{\gamma_1\left(\Delta\sigma r_{HL} + \Delta\theta q_{HH}\right)\right.\\
&+\gamma_2\left[\beta\Delta\theta q_{HL} + (1-\beta)\left(\Delta\sigma r_{HL} + \Delta\theta q_{HH} - \Delta\sigma r_{HH}\right) + \Delta\sigma r_{LL}\right]\\
&\left.+\gamma_3\left(\Delta\theta q_{HL} + \Delta\sigma r_{HL}\right)\right\}.
\end{aligned}$$

Optimization with respect to quantities yields the first-order conditions listed in Lemma 4.

A.3 Proof of Lemma 5

Denote

$$\phi(\beta) \equiv \frac{\frac{1}{\nu} - (\beta-\mu)\left(\gamma_2 + \frac{\mu}{1-\mu}\right)}{1 - (\gamma_1+\gamma_2) + \gamma_2\beta - \frac{(1-\beta)\mu}{1-\mu}}.$$

The numerator of $\phi(\beta)$ is strictly positive so that $\phi(\beta) \neq 0$. The denominator is either positive or negative depending upon the value of β. Suppose that β is such that $\phi(\beta) > 0$. Then, from

Lemma 4, $\underline{q}_{HH} > \underline{q}_{HL}$ if and only if

$$\frac{\Delta\theta}{\Delta\sigma} > -\phi(\beta), \tag{38}$$

which is true. Moreover, $\overline{q}_{HL} > \overline{q}_{HH}$ if and only if $\phi(\beta) > 1$ and

$$\frac{\Delta\theta}{\Delta\sigma} < \phi(\beta). \tag{39}$$

Suppose now that $\phi(\beta) < 0$. Then, from Lemma 4, $\overline{q}_{HL} > \overline{q}_{HH}$ and so, equivalently,

$$\frac{\Delta\theta}{\Delta\sigma} > \phi(\beta). \tag{40}$$

Moreover, $\underline{q}_{HH} > \underline{q}_{HL}$ if and only if $-\phi(\beta) > 1$ and

$$\frac{\Delta\theta}{\Delta\sigma} < -\phi(\beta). \tag{41}$$

Therefore, when $\frac{\Delta\theta}{\Delta\sigma} < |\phi(\beta)|$ or, equivalently, (14) holds, the quantity solution in Lemma 4 satisfies (38) and (39) for $\phi(\beta) > 0$, and (40) and (41) for $\phi(\beta) < 0$. We thus have $\underline{q}_{HH} > \underline{q}_{HL}$ together with $\overline{q}_{HL} > \overline{q}_{HH}$.

Consider now the case of $\frac{\Delta\theta}{\Delta\sigma} \geq |\phi(\beta)|$. Take first $\phi(\beta) > 0$. Then, (38) is satisfied but (39) is not so that, under Lemma 4, $\underline{q}_{HH} > \underline{q}_{HL}$ and $\overline{q}_{HL} \leq \overline{q}_{HH}$. It follows that (16) is violated with strict inequality:

$$(\Delta\theta + \Delta\sigma)(\overline{q}_{HL} - \overline{q}_{HH}) < (\Delta\theta - \Delta\sigma)(\underline{q}_{HH} - \underline{q}_{HL}). \tag{42}$$

However, this is equivalent to $\Pi_{LL,1} < \Pi_{LL,2}$ and so to $\beta = 0$. As $\phi(0) < 0$, the initial hypothesis leads to a contradiction. Take now $\phi(\beta) < 0$. Then, (40) holds but (41) does not, meaning that $\overline{q}_{HL} > \overline{q}_{HH}$ and $\underline{q}_{HH} \leq \underline{q}_{HL}$. It implies that (16) is satisfied, meaning that $\Pi_{LL,1} > \Pi_{LL,2}$ and so $\beta = 1$. However, $\phi(1) > 0$, which contradicts the initial hypothesis. Therefore, when $\frac{\Delta\theta}{\Delta\sigma} \geq |\phi(\beta)|$ or, equivalently, (14) is violated, β is such that $\frac{\Delta\theta}{\Delta\sigma} = |\phi(\beta)|$, implying that $\underline{q}_{HH} = \underline{q}_{HL}$ and $\overline{q}_{HL} = \overline{q}_{HH}$.

A.4 Proof of Lemma 6

Under the assumption that S' is very concave, it follows from Lemma 7 that $\beta = 1$. From Lemma 4, for $\beta = 1$ we find:

$$\begin{aligned} S'(\underline{q}_{HL}) - S'(\underline{q}_{LH}) &= \Delta\theta + \Delta\sigma \\ &+ \frac{\nu}{1-\nu}\left\{\left[1 + (\gamma_2+\gamma_3)\frac{1-\mu}{\mu}\right]\Delta\theta + (1-\nu\gamma_2)\frac{1-\mu}{\nu\mu}\Delta\sigma\right\} \end{aligned} \tag{43}$$

$$\begin{aligned} S'(\overline{q}_{HL}) - S'(\overline{q}_{LH}) &= \Delta\theta - \Delta\sigma \\ &+ \frac{\nu}{1-\nu}\left\{\left[1 + (\gamma_2+\gamma_3)\frac{1-\mu}{\mu}\right]\Delta\theta - (1-\nu\gamma_2)\frac{1-\mu}{\nu\mu}\Delta\sigma\right\} \end{aligned} \tag{44}$$

Thus, $\underline{q}_{LH} > \underline{q}_{HL}$. First suppose that $\overline{q}_{HL} < \overline{q}_{LH}$, which is equivalent to $\frac{\Delta\theta}{\Delta\sigma} > \delta$. Then, $r_{LH} \geq r_{HL}$ if and only if S' is not very concave. Suppose now that $\overline{q}_{HL} > \overline{q}_{LH}$, which is equivalent to $\frac{\Delta\theta}{\Delta\sigma} < \delta$. Then, $q_{LH} \geq q_{HL}$ if and only if S' is not very concave

Hence, when $\frac{\Delta\theta}{\Delta\sigma} \neq \delta$ at the solution, and S' is very concave, either $q_{LH} \geq q_{HL}$ or $r_{LH} \geq r_{HL}$ is violated. The proof of Solution 1 below shows that these conditions are both necessary for the incentive constraints in Γ to be all satisfied. Hence, the quantities assigned to HL and LH are not as pinned down in Lemma 4. From the proof of Solution 1 below, when S' is very concave, $r_{LH} = r_{HL}$ for $\frac{\Delta\theta}{\Delta\sigma} > \delta$ and $q_{LH} = q_{HL}$ for $\frac{\Delta\theta}{\Delta\sigma} < \delta$.

B Proof of Lemma 7

From Lemma 5, with (14) satisfied, $\underline{q}_{HH} > \underline{q}_{HL}$ together with $\overline{q}_{HL} > \overline{q}_{HH}$. Using Lemma 4, we calculate

$$[S'(\underline{q}_{HL}) - S'(\underline{q}_{HH})] - [S'(\overline{q}_{HH}) - S'(\overline{q}_{HL})] = 2\left(\beta\gamma_2 + \gamma_3 - \frac{(1-\beta)\,\mu}{1-\mu}\right)\frac{\nu}{(1-\nu)\,\mu}\Delta\theta. \quad (45)$$

We look for the conditions under which (15) is satisfied and so $\beta = 1$. Replacing $\beta = 1$, (45) is rewritten as

$$[S'(\underline{q}_{HL}) - S'(\underline{q}_{HH})] - [S'(\overline{q}_{HH}) - S'(\overline{q}_{HL})] = 2\frac{\nu\,(\gamma_2+\gamma_3)}{\mu\,(1-\nu)}\Delta\theta, \quad (46)$$

from which we deduce that, as long as $\gamma_2 + \gamma_3 > 0$, $\overline{q}_{HL} - \overline{q}_{HH} > \underline{q}_{HH} - \underline{q}_{HL}$ (equivalently, $q_{HL} > q_{HH}$) if and only if S' is sufficiently concave. Moreover, the difference $q_{HH} - q_{HL}$ increases as S' becomes less concave/more convex. With $q_{HL} > q_{HH}$, (15) holds as long as S' is sufficiently concave. The proof of Solution 5 below shows that (15) holds as well when S' is (almost) linear and the difference $q_{HH} - q_{HL}$ is positive but small. (15) is violated for S' sufficiently convex that the difference $q_{HH} - q_{HL}$ is large enough to violate (16). Finally, when S' is sufficiently concave and $q_{HL} > q_{HH}$, it follows from Lemma 3 that $\gamma_3 > 0$. The proof of Solution 5 shows that $\gamma_1 < 1$ for S' not sufficiently convex, which confirms that $\gamma_2 + \gamma_3 > 0$.

C Case 1

C.1 Proof of Solution 1

From Lemma 7, $\beta = 1$. Using Lemma 3 and $\beta = 1$, we deduce that $\gamma_3 > 0$ if and only if $q_{HL} \geq q_{HH}$ and $r_{LL} \geq r_{HL}$, with $\gamma_3 = 1$ if the two inequalities hold strictly. From Lemma 7, $q_{HL} > q_{HH}$ when S' is sufficiently concave and (14) is satisfied. Using Lemma 4, we find for $\beta = 1$:

$$S'(\underline{q}_{HL}) - S'(\underline{q}_{LL}) = \left\{1 + \frac{\nu}{1-\nu}\left[1 + \frac{1-\mu}{\mu}(\gamma_2+\gamma_3)\right]\right\}\Delta\theta + \frac{(1-\gamma_2)(1-\mu)}{\mu(1-\nu)}\Delta\sigma \quad (47)$$

$$S'(\overline{q}_{HL}) - S'(\overline{q}_{LL}) = \left\{1 + \frac{\nu}{1-\nu}\left[1 + \frac{1-\mu}{\mu}(\gamma_2+\gamma_3)\right]\right\}\Delta\theta - \frac{(1-\gamma_2)(1-\mu)}{\mu(1-\nu)}\Delta\sigma \quad (48)$$

Hence, $\underline{q}_{LL} > \underline{q}_{HL}$. Moreover, $\overline{q}_{LL} \geq \overline{q}_{HL}$ if and only if

$$\frac{\Delta\theta}{\Delta\sigma} \geq \frac{\gamma_2 (1-\mu)}{\mu + \gamma_1 \nu (1-\mu)}. \tag{49}$$

Supposing that (49) holds and computing

$$[S'(\underline{q}_{HL}) - S'(\underline{q}_{LL})] - [S'(\overline{q}_{HL}) - S'(\overline{q}_{LL})] = 2\frac{(1-\gamma_2)(1-\mu)}{\mu(1-\nu)}\Delta\sigma, \tag{50}$$

we deduce that, when $\gamma_1 + \gamma_3 > 0$ and (49) holds, $r_{HL} > r_{LL}$ if and only if S' is sufficiently concave. When (49) does not hold, $r_{LL} > r_{HL}$ whatever the shape of S'.

Overall, when both (14) and (49) hold, it is $q_{HL} > q_{HH}$ together with $r_{HL} > r_{LL}$, meaning that $\gamma_3 = 1$. Substituting $\gamma_3 = 1$ into (14) and (49) yields

$$\frac{\Delta\theta}{\Delta\sigma} \in \left(\frac{1-\mu}{\mu + \nu(1-\mu)}, \frac{1-\nu\mu}{\nu}\right). \tag{51}$$

When $\frac{\Delta\theta}{\Delta\sigma} \geq \frac{1-\nu\mu}{\nu}$ (14) does not hold for $\gamma_3 = 1$. Suppose that it does hold for some $\gamma_3 < 1$. (49) is satisfied, implying that $r_{HL} > r_{LL}$ and so $\gamma_2 = 0$. Moreover, the proof of Solution 5 below shows that $\gamma_1 \neq 1$ whenever S' is concave. Thus, when $\frac{\Delta\theta}{\Delta\sigma} \geq \frac{1-\nu\mu}{\nu}$ and, at the same time, (14) holds, $\gamma_3 > 0$ and $\gamma_1 > 0$ so that $q_{HL} = q_{HH}$. (14) is rewritten as $\frac{\Delta\theta}{\Delta\sigma} < \frac{1-\nu\mu}{\nu\gamma_3}$ and it is feasible with $\frac{\Delta\theta}{\Delta\sigma} \geq \frac{1-\nu\mu}{\nu}$.

When $\frac{\Delta\theta}{\Delta\sigma} \leq \frac{1-\mu}{\mu+\nu(1-\mu)}$, (49) is violated for $\gamma_3 = 1$. If (14) holds, then $q_{HL} > q_{HH}$ and so $\gamma_1 = 0$. Moreover, the proof of Solution 5 below shows that $\gamma_2 = 1$ only if S' is convex. Hence, $\gamma_3 > 0$ and $\gamma_2 > 0$, meaning that $r_{HL} = r_{LL}$.

C.1.1 Check the incentive constraints in Γ

From S' concave and (14) satisfied, it follows from Lemma 7 that (15) holds. Hence (IC1) is binding and (IC3) is slack. (IC2) is rewritten $r_{LH} \geq r_{HL}$, which is satisfied if $\frac{\Delta\theta}{\Delta\sigma} \leq \delta$; otherwise is satisfied when S' is not very concave (see Lemma 6). (IC4) is rewritten $q_{LL} \geq q_{HL}$. From (48), $\underline{q}_{LL} > \underline{q}_{HL}$. If (49) holds, then $\overline{q}_{LL} > \overline{q}_{HL}$ and (IC4) is satisfied. If (49) does not hold, then $q_{LL} \geq q_{HL}$ and so (IC4) is satisfied if and only if S' is not very concave. (IC5) is equivalent to $r_{HH} \geq r_{HL}$, which is satisfied because $\underline{q}_{HH} > \underline{q}_{HL}$ and, since (14) holds by assumption, $\overline{q}_{HL} > \overline{q}_{HH}$. (IC6) is rewritten $\Delta\theta (q_{LH} - q_{HL}) \geq \Delta\sigma (r_{HL} - r_{LH})$, which is satisfied when so are (IC2) and (IC10). (IC7) and (IC8) are equivalent to $q_{HL} \geq q_{HH}$ and $r_{HL} \geq r_{LL}$, both being satisfied because $\gamma_3 > 0$. (IC10) is rewritten as $q_{LH} \geq q_{HL}$, which is satisfied if $\frac{\Delta\theta}{\Delta\sigma} \geq \delta$; otherwise is satisfied when S' is not very concave (see Lemma 6). (IC11) is satisfied because it was assumed to be binding in deriving quantities of Lemma 4. (IC12) is equivalent to $\Delta\theta q_{LL} + \Delta\sigma r_{HL} \geq \Delta\theta q_{HL} + \Delta\sigma r_{LL}$, and is satisfied when so are (IC4) and (IC8).

C.1.2 Quantity solution when (IC2), (IC4) or (IC10) is violated

From Lemma 4 we have

$$S'(\overline{q}_{LH}) - S'(\overline{q}_{LL}) = \left(1 + \gamma_2 \frac{1-\mu}{\mu}\right)\Delta\sigma, \tag{52}$$

from which $\overline{q}_{LL} > \overline{q}_{LH}$.

Suppose first that $\frac{\Delta\theta}{\Delta\sigma} \in \left(\frac{1-\mu}{\mu+\nu(1-\mu)}, \frac{1-\nu\mu}{\nu}\right)$ (scenario (ii) in Solution 1). We previously found that $\gamma_3 = 1$ and that $\overline{q}_{HL} > \overline{q}_{HH}$ ((14) satisfied). Replacing $\gamma_3 = 1$ into (17), we find $\delta = \frac{1-\nu\mu}{\mu+\nu(1-\mu)}$, which belongs to $\left(\frac{1-\mu}{\mu+\nu(1-\mu)}, \frac{1-\nu\mu}{\nu}\right)$.

Take $\frac{\Delta\theta}{\Delta\sigma} \in \left(\frac{1-\nu\mu}{\mu+\nu(1-\mu)}, \frac{1-\nu\mu}{\nu}\right)$ *i.e.*, $\frac{\Delta\theta}{\Delta\sigma} > \delta$, so that $\overline{q}_{LH} > \overline{q}_{HL}$. Then, the bad-state quantities in Lemma 4 are ranked as

$$\overline{q}_{LL} > \overline{q}_{LH} > \overline{q}_{HL} > \overline{q}_{HH}. \tag{53}$$

Suppose that (53) holds at the solution to Γ. Then, (IC4) and (IC10) are slack, (IC2) is binding. Then, the most natural guess is that (IC2) is binding together with the constraints that are binding in Γ'. In particular, the binding constraint (IC1) is rewritten as $r_{HL} = r_{LH}$. Reformulating the problem as the maximization of Γ' subject to $r_{HL} = r_{LH}$, and associating the multiplier $\lambda_2 > 0$ to this constraint, the optimal quantities of types LH and HL are pinned down as in $(23) - (25)$ for $\gamma_3 = 1$. For types HH and LL quantities are still those in Lemma 4 for $\beta = 1$ and $\gamma_3 = 1$. For types LH and HL production levels are such that r_{HL} is decreased below and r_{LH} raised above the respective levels under Lemma 4. Specifically, $\overline{q}_{LH}$ is decreased and $\overline{q}_{HL}$ raised. Moreover, because $\underline{q}_{LH} > \underline{q}_{HL}$ at optimum, and λ_2 is such that $r_{HL} = r_{LH}$, it follows that it is still $\overline{q}_{LH} > \overline{q}_{HL}$ at optimum. Therefore, (53) is still the ranking of optimal bad-state quantities. Checking again the incentive constraints of Γ, it follows trivially that they are all satisfied.

Take now $\frac{\Delta\theta}{\Delta\sigma} \in \left(\frac{1-\mu}{\mu+\nu(1-\mu)}; \frac{1-\nu\mu}{\mu+\nu(1-\mu)}\right)$ *i.e.*, $\frac{\Delta\theta}{\Delta\sigma} < \delta$, so that $\overline{q}_{HL} > \overline{q}_{LH}$. Then, the bad-state quantities in Lemma 4 are ranked as

$$\overline{q}_{LL} > \overline{q}_{HL} > \overline{q}_{LH} > \overline{q}_{HH}. \tag{54}$$

As above, suppose that (54) holds at the solution to Γ. Then, (IC2) and (IC4) are slack, (IC10) is binding. Then, the most natural guess is that (IC10) is binding together with the constraints that are binding in Γ'. In particular, the binding constraint (IC11) is rewritten as $q_{HL} = q_{LH}$. Reformulating the problem as the maximization of Γ' subject to $q_{HL} = q_{LH}$, and associating the multiplier $\lambda_1 > 0$ to this constraint, the optimal quantities are pinned down as in $(19) - (21)$ for $\gamma_3 = 1$. For types HH and LL quantities are still those in Lemma 4. For types LH and HL production levels are such that q_{HL} is decreased below and q_{LH} raised above the respective levels in Lemma 4. Hence, $\overline{q}_{LH}$ is raised and $\overline{q}_{HL}$ is decreased. Because, at optimum, $\underline{q}_{LH} > \underline{q}_{HL}$, and λ_1 is such that $q_{HL} = q_{LH}$, it is still $\overline{q}_{HL} > \overline{q}_{LH}$. Therefore, (54) is still the ranking of optimal bad-state quantities. Checking again the incentive constraints in Γ, it follows trivially that they are all satisfied.

Suppose now that $\frac{\Delta\theta}{\Delta\sigma} \geq \frac{1-\nu\mu}{\nu}$ (scenario (iii) in Solution 1). We know that now $\gamma_3 > 0$, $\gamma_1 > 0$ so that $q_{HL} = q_{HH}$, and $\gamma_2 = 0$. To have $q_{HL} = q_{HH}$, (14) must still be valid so that $\overline{q}_{HL} > \overline{q}_{HH}$. This is equivalent to $\frac{\Delta\theta}{\Delta\sigma} < \frac{1-\nu\mu}{\nu\gamma_3}$. Hence, the interval $\left[\frac{1-\nu\mu}{\nu}, \frac{1-\nu\mu}{\nu\gamma_3}\right)$ does exist, and the magnitude of γ_3 is such that $q_{HL} = q_{HH}$. Replacing $\gamma_2 = 0$, (17) becomes $\delta = \frac{1-\nu\mu}{\mu+\gamma_3\nu(1-\mu)}$. With $\frac{\Delta\theta}{\Delta\sigma} \in \left[\frac{1-\nu\mu}{\nu}, \frac{1-\nu\mu}{\nu\gamma_3}\right)$, both $\frac{\Delta\theta}{\Delta\sigma} < \delta$ and $\frac{\Delta\theta}{\Delta\sigma} > \delta$ are feasible.

First take $\frac{\Delta\theta}{\Delta\sigma} > \delta$, in which case $\overline{q}_{LH} > \overline{q}_{HL}$ under Lemma 4. Recalling also from (52) that $\overline{q}_{LL} > \overline{q}_{LH}$, the bad-state quantities are ranked as in (53). As above, supposing that (53) holds at the solution to Γ, the optimal quantities of types LH and HL are pinned down as in $(23) - (26)$, but with $\gamma_2 = 0$ in this case. For types HH and LL, quantities are still pinned down as in Lemma 4 with $\beta = 1$ and $\gamma_2 = 0$. The production levels of types LH and HL are such that r_{HL} is decreased below and r_{LH} raised above the respective levels under Lemma 4. Specifically, $\overline{q}_{LH}$ is decreased and $\overline{q}_{HL}$ raised. Moreover, because $\underline{q}_{LH} > \underline{q}_{HL}$ at optimum, and λ_2 is such that $r_{HL} = r_{LH}$, it follows that it is still $\overline{q}_{LH} > \overline{q}_{HL}$ at optimum. Therefore, (53) is still the ranking of optimal bad-state quantities. Checking again the incentive constraints in Γ, it follows trivially that they are all satisfied.

Second take $\frac{\Delta\theta}{\Delta\sigma} < \delta$, so that $\overline{q}_{HL} > \overline{q}_{LH}$. Then, the bad-state quantities in Lemma 4 are ranked as in (54). As above, supposing that (54) holds at the solution to Γ, the optimal quantities of types HH and LL are still those pinned down in Lemma 4 with $\beta = 1$ and $\gamma_2 = 0$. The production levels of types LH and HL are such that q_{HL} is decreased below and q_{LH} raised above the respective levels under Lemma 4. Specifically, $\overline{q}_{LH}$ is raised and $\overline{q}_{HL}$ is decreased, as indicated by $\lambda_1 > 0$ in (19) – (21). Because, at optimum, $\underline{q}_{LH} > \underline{q}_{HL}$, and λ_1 is such that $q_{HL} = q_{LH}$, it is still $\overline{q}_{HL} > \overline{q}_{LH}$. Therefore, (54) is still the ranking of optimal bad-state quantities. Checking again the incentive constraints in Γ, it follows trivially that they are all satisfied.

Lastly, suppose that $\frac{\Delta\theta}{\Delta\sigma} \leq \frac{1-\mu}{\mu+\nu(1-\mu)}$ (scenario (i) in the Solution 1). In this case, $\gamma_3 > 0$, $\gamma_2 > 0$ and $\gamma_1 = 0$ so that $r_{HL} = r_{LL}$. Then, (49) is necessarily satisfied so that $\overline{q}_{LL} > \overline{q}_{HL}$. The range of values of $\frac{\Delta\theta}{\Delta\sigma}$ for which $\frac{\Delta\theta}{\Delta\sigma} \leq \frac{1-\mu}{\mu+\nu(1-\mu)}$ and, simultaneously, (14) and (49) hold, is identified as

$$\left[\frac{\gamma_3(1-\mu)}{\mu+\nu(1-\mu)}, \min\left\{\frac{1-\nu[1-\gamma_3(1-\mu)]}{\nu}; \frac{1-\mu}{\mu+\nu(1-\mu)}\right\}\right].$$

With $\gamma_1 = 0$ (17) becomes $\delta \equiv \frac{1-\nu[1-\gamma_3(1-\mu)]}{\mu+\nu(1-\mu)}$. As long as $\frac{\Delta\theta}{\Delta\sigma}$ is drawn on the above range, both $\frac{\Delta\theta}{\Delta\sigma} < \delta$ and $\frac{\Delta\theta}{\Delta\sigma} > \delta$ are feasible. First take $\frac{\Delta\theta}{\Delta\sigma} > \delta$ and so $\overline{q}_{HL} < \overline{q}_{LH}$. Then, once more, the solution is as found under (53), here with $\beta = 1$, $\gamma_1 = 0$, and λ_2 such that $r_{HL} = r_{LL}$. Next take $\frac{\Delta\theta}{\Delta\sigma} < \delta$. Then, the solution is as found under (54), here with $\beta = 1$, $\gamma_1 = 0$, and λ_1 such that $q_{HL} = q_{LL}$.

C.2 Proof of Solution 2

Recall from Solution 1 that, when (14) holds at the solution to Γ, and S' is sufficiently concave, under Lemma 4, either $q_{HL} > q_{HH}$ or $r_{HL} > r_{LL}$, or both, so that $\gamma_3 > 0$. We also found that, for S' very concave, the quantities in Lemma 4 do not satisfy either $q_{LH} \geq q_{HL}$ or $r_{LH} \geq r_{HL}$, and the production levels of types LH and HL are not pinned down as in Lemma 4. They are such that either $q_{HL} = q_{LH}$ or $r_{HL} = r_{LH}$. Based on Lemma 4, for $\beta = 1$ we compute

$$S'(\underline{q}_{HH}) - S'(\underline{q}_{LH}) = S'(\overline{q}_{HH}) - S'(\overline{q}_{LH}) = \left(1 + \gamma_1 \frac{\nu}{1-\nu}\right)\Delta\theta, \tag{55}$$

from which $q_{LH} > q_{HH}$. This inequality is still valid in Solution 1, provided in Solution 1 q_{LH} is raised relative to Lemma 4 $(\lambda_1 > 0)$. Therefore, we cannot have a solution in which $q_{HL} = q_{LH}$ together with $q_{HH} \geq q_{HL}$. When $q_{HL} > q_{HH}$ in Solution 1 (scenarios (ii) and (iii)), we have $q_{LH} = q_{HL} > q_{HH}$ for S' very concave, and $q_{LH} > q_{HL} > q_{HH}$ for S' sufficiently (but not very) concave.

Under Lemma 4, we further have

$$S'(\underline{q}_{LL}) - S'(\underline{q}_{LH}) = \left(1 + \gamma_2 \frac{1-\mu}{\mu}\right)\Delta\sigma, \tag{56}$$

from which $\underline{q}_{LH} > \underline{q}_{LL}$. Because it is also $\overline{q}_{LL} > \overline{q}_{LH}$ (as from 52), $r_{LH} > r_{LL}$ under Lemma 4. Moreover, when $r_{HL} = r_{LH}$ in Solution 1, r_{LH} is raised relative to Lemma 4 $(\lambda_2 > 0)$, and it is still $r_{LH} > r_{LL}$. Therefore, we cannot have a solution in which $r_{HL} = r_{LH}$ together with $r_{LL} \geq r_{HL}$. When $r_{HL} > r_{LL}$ in Solution 1 (scenarios (i) and (ii)), we have $r_{LH} = r_{HL} > r_{LL}$

for S' very concave, and $r_{LH} > r_{HL} > r_{LL}$ for S' sufficiently (but not very) concave.

Overall, when S' is sufficiently concave to have $r_{HL} > r_{LL}$ for the values of $\frac{\Delta\theta}{\Delta\sigma}$ in (i) and (ii) of Solution 1, and $q_{HL} > q_{HH}$ for the values of $\frac{\Delta\theta}{\Delta\sigma}$ in (ii) and (iii) of Solution 1, but not very concave as required in Solution 1, so that $q_{LH} \geq q_{HL}$ and $r_{LH} \geq r_{HL}$ under Lemma 4, the solution is pinned down as in Solution 1, but with $\lambda_1 = \lambda_2 = 0$.

C.3 Proof of Solution 3

Take (14) to hold and $\frac{\Delta\theta}{\Delta\sigma} \in \left(\frac{1-\mu}{\mu+\nu(1-\mu)}, \frac{1-\nu\mu}{\nu}\right)$. Further take S' sufficiently concave that, at optimum, $q_{HL} > q_{HH}$ together with $r_{HL} > r_{LL}$, as in Solution 1 and 2. Recall that, in these situations, $\gamma_3 = 1$. First consider $q_{HL} - q_{HH}$. From (46) ,, keeping $\gamma_2 + \gamma_3$ fixed, this difference decreases as S' becomes less concave. At some little degree of concavity of S', the condition $q_{HL} > q_{HH}$ stops holding. Next consider $r_{HL} - r_{LL}$. From (50) ,, keeping $\gamma_1 + \gamma_3$ fixed, this difference decreases as S' becomes less concave. At some little degree of concavity of S', the condition $r_{HL} > r_{LL}$ stops holding. Hence, with $\gamma_3 = 1$ (as in in Solution 1), on decreasing degrees of concavity of S', quantities in Lemma 4 indicate that there exists a solution in which one of the inequalities $q_{HL} > q_{HH}$ and $r_{HL} > r_{LL}$ is not satisfied and then, as S' becomes even less concave, both of them are not satisfied.

Suppose that $\frac{\Delta\theta}{\Delta\sigma}$ tends to $\frac{1-\mu}{\mu+\nu(1-\mu)}$ from above, meaning that $\overline{q}_{LL}$ comes very close to $\overline{q}_{HL}$, and so $r_{HL} - r_{LL}$ is negative in the limit. On the other hand, $q_{HL} > q_{HH}$ for S' sufficiently concave. Next suppose that $\frac{\Delta\theta}{\Delta\sigma}$ tends to $\frac{1-\nu\mu}{\nu}$ from below, meaning that $\overline{q}_{HH}$ comes very close to $\overline{q}_{HL}$ and so $q_{HL} - q_{HH}$ is negative in the limit. On the other hand, $r_{HL} > r_{LL}$ for S' sufficiently concave. Then, resting on (46) and (50) , the difference $q_{HL} - q_{HH}$ decreases as $\Delta\theta$ increases, and is independent of $\Delta\sigma$; the difference $r_{HL} - r_{LL}$ decreases as $\Delta\sigma$ increases, and is independent of $\Delta\theta$. Hence, $\exists \delta_1 \in \left(\frac{1-\mu}{\mu+\nu(1-\mu)}, \frac{1-\nu\mu}{\nu}\right)$ such that, for some less than "sufficiently" concave S', $r_{LL} \geq r_{HL}$ and $q_{HL} > q_{HH}$ for $\frac{\Delta\theta}{\Delta\sigma} < \delta_1$, whereas $r_{HL} > r_{LL}$ and $q_{HH} \geq q_{HL}$ for $\frac{\Delta\theta}{\Delta\sigma} > \delta_1$.

We now show that, for the degrees of concavity of S' considered above, $r_{LL} = r_{HL}$ when $\frac{\Delta\theta}{\Delta\sigma} < \delta_1$ (as in scenario (iii) of Solution 1) and $q_{HH} = q_{HL}$ when $\frac{\Delta\theta}{\Delta\sigma} > \delta_1$ (as in scenario (i) of Solution 1).

First suppose that $\frac{\Delta\theta}{\Delta\sigma} > \delta_1$. When S' is such that $q_{HL} = q_{HH}$ for $\gamma_3 = 1$, according to (46) , one could still have $q_{HL} > q_{HH}$ if it were $\gamma_3 < 1$. However, when $q_{HL} > q_{HH}$, γ_3 must be 1, provided $r_{HL} > r_{LL}$. It means that $q_{HL} \leq q_{HH}$ for any lower degree of concavity of S'. Let us now focus on $q_{HL} = q_{HH}$. As S' becomes less concave, according to (46) , one could still have $q_{HL} = q_{HH}$ with a smaller value of $\gamma_2 + \gamma_3$. As $\gamma_2 = 0$ (with $r_{HL} > r_{LL}$), this would mean to have $\gamma_3 < 1$ and $\gamma_1 > 0$. Having $q_{HL} = q_{HH}$ means that $\gamma_3 > 0$ and $\gamma_1 > 0$. Hence, there exists some range of degrees of concavity of S' for which $q_{HL} = q_{HH}$.

Next suppose that $\frac{\Delta\theta}{\Delta\sigma} < \delta_1$. Resting on (50) (as on (46) above), we conclude that, for $r_{HL} = r_{LL}$ to be attained for some degrees of concavity of S' for which $q_{HL} > q_{HH}$, γ_3 must decrease below 1 and γ_2 increase above 0 by an equal amount.

Overall, the conditions $r_{HL} = r_{LL}$ $(\gamma_1 > 0, \gamma_3 > 0)$ and $q_{HL} > q_{HH}$ $(\gamma_2 = 0)$, which were found to hold for $\frac{\Delta\theta}{\Delta\sigma} < \frac{1-\mu}{\mu+\nu(1-\mu)}$ (scenario (i) in Solution 1), also hold for $\frac{\Delta\theta}{\Delta\sigma} < \delta_1$, with $\delta_1 > \frac{1-\mu}{\mu+\nu(1-\mu)}$. Moreover, the conditions $q_{HL} = q_{HH}$ $(\gamma_1 > 0, \gamma_3 > 0)$ and $r_{HL} > r_{LL}$ $(\gamma_1 = 0)$, which were found to hold for $\frac{\Delta\theta}{\Delta\sigma} > \frac{1-\nu\mu}{\nu}$ (scenario (iii) in Solution 1), also holds for $\frac{\Delta\theta}{\Delta\sigma} > \delta_1$, with $\delta_1 < \frac{1-\nu\mu}{\nu}$.

One can prove that all incentive constraints in Γ are satisfied following the same procedure as for Solution 1, yet with $\lambda_1 = \lambda_2 = 0$ as in Solution 2.

C.4 Proof of Solution 4

Start from Solution 3 and $\frac{\Delta\theta}{\Delta\sigma} < \delta_1$ so that $r_{LL} = r_{HL}$ and $q_{HL} > q_{HH}$. From the proof of Solution 3 we know that, as S' becomes less concave, for $r_{LL} = r_{HL}$ to be kept, there must be a decrease in γ_3 and an equal increase in γ_2 that leave $\gamma_3 + \gamma_2$ and, hence, the right-hand side of (46) unchanged. This implies that $q_{HL} - q_{HH}$ decreases. At the limit, if it is still $q_{HL} > q_{HH}$, then $\gamma_3 = 0$ and $\gamma_2 = 1$. However, the proof of Solution 5 below shows that $\gamma_2 = 1$ only if S' is convex. Hence, for some degree of concavity of S', at the solution to Γ, $r_{LL} = r_{HL}$ together with $q_{HL} \leq q_{HH}$.

We now show that there exists a solution in which $r_{LL} = r_{HL}$ together with $q_{HL} = q_{HH}$. Start from the highest degree of concavity of S' for which $q_{HL} = q_{HH}$. Based on (46), we see that, as S' becomes slightly less concave, for $q_{HL} = q_{HH}$ to be kept, it must be the case that $\gamma_2 + \gamma_3$ decreases (and γ_1 increases) by some $\varepsilon > 0$. On the other hand, based on (50), we see that, for $r_{LL} = r_{HL}$ to be kept, it must be the case that $\gamma_1 + \gamma_3$ decreases (and γ_2 increases) by some $\zeta > 0$. Overall, it must be the case that γ_1 increases by ε and γ_2 by ζ, and that $\gamma_2 + \gamma_3$ decreases by ε and $\gamma_1 + \gamma_3$ by ζ. Denoting d_3 the change in γ_3, the following conditions must hold:

$$\begin{aligned}\gamma_2 + \zeta + \gamma_3 + d_3 &= \gamma_2 + \gamma_3 - \varepsilon \\ \gamma_1 + \varepsilon + \gamma_3 + d_3 &= \gamma_1 + \gamma_3 - \zeta.\end{aligned}$$

Hence, it must be $d_3 = -(\varepsilon + \zeta)$. That is, γ_3 must decrease by $\varepsilon + \zeta$. All this requires that the values of γ_2 and γ_3 in the initial situation and the values of ε and ζ be such that $\gamma_2 + \zeta < 1$ together with $\gamma_3 > \varepsilon + \zeta$. This is feasible for a very small decrease in the degree of concavity of S'. Then, at the solution to Γ, $q_{HL} = q_{HH}$ together with $r_{LL} = r_{HL}$.

The proof for the case of $\frac{\Delta\theta}{\Delta\sigma} > \delta_1$ proceeds similarly to the one developed above.

One can prove that all incentive constraints in Γ are satisfied following the same procedure as for Solution 1, yet with $\lambda_1 = \lambda_2 = 0$ as in Solution 2.

D Case 2

D.1 Proof of Solution 5

Starting from the proof of Solution 4, as the degree of concavity of S' becomes little enough, at least one of conditions $r_{LL} = r_{HL}$ and $q_{HL} = q_{HH}$ becomes unfeasible. Then, either $q_{HL} < q_{HH}$ or $r_{LL} < r_{HL}$ or both. In any such situation, $\gamma_3 = 0$ and we move to Case 2. Then, either γ_1 and γ_2 are both positive or only one of them is. We hereafter show that, for S' almost linear, $\gamma_1 < 1$ and $\gamma_2 < 1$. We first suppose that, at the solution, $\gamma_1 = 1$ meaning that $q_{HH} > q_{HL}$ together with

$$\Delta\theta(q_{HH} - q_{HL}) > \Delta\sigma(r_{LL} - r_{HL}). \tag{57}$$

From (46), with $\gamma_1 = 1$, $q_{HH} > q_{HL}$ if and only if S' is strictly convex. Hence, $\gamma_1 = 1$ only if S' is strictly convex. Moreover, from (50), with $\gamma_1 = 1$, $r_{LL} > r_{HL}$ as long as S' is convex. Then, (57) holds only for S' sufficiently convex. We next suppose that, at the solution, $\gamma_2 = 1$ meaning that $r_{LL} > r_{HL}$ together with

$$\Delta\sigma(r_{LL} - r_{HL}) > \Delta\theta(q_{HH} - q_{HL}). \tag{58}$$

Then, from (50), $r_{LL} > r_{HL}$ if and only if S' is strictly convex. Hence, $\gamma_2 = 1$ only if S' is strictly convex. From (46), with $\gamma_2 = 1$, $q_{HH} > q_{HL}$ as long as S' is convex. Then, (58) holds only for S' sufficiently convex. Hence, for S' almost linear, $\gamma_1 > 0$ and $\gamma_2 > 0$.

D.1.1 Check incentive constraints in Γ

(IC1) is binding and (IC3) is slack because $\beta = 1$ (as shown above). Hence, they are both satisfied. With (IC8) binding $(\gamma_2 > 0)$, (IC2) is rewritten as $r_{LH} \geq r_{LL}$. From (52) and (56), $\overline{q}_{LL} > \overline{q}_{LH}$ and $\underline{q}_{LH} > \underline{q}_{LL}$ so that (IC2) is slack. Further using the binding constraints (IC8) in (IC4), the latter is rewritten as $q_{LL} \geq q_{HL}$. From 61, $\overline{q}_{LL} > \overline{q}_{HH}$; also, from the proof of Lemma 8 below, $\underline{q}_{LL} > \underline{q}_{HH}$. Hence, $q_{LL} \geq q_{HH}$. As $q_{HH} > q_{HL}$ (see above), (IC4) is slack. With (IC11) binding (Lemma 3), (IC5) is rewritten as $r_{HH} \geq r_{HL}$. Recall that (14) is assumed to hold. It follows from Lemma 5 that $\underline{q}_{HH} > \underline{q}_{HL}$ and $\overline{q}_{HL} > \overline{q}_{HH}$ so that (IC5) is slack. Using the binding constraints (IC8) and (IC1) in (IC6), the latter becomes $\Delta\theta\,(q_{LH} - q_{HL}) \geq \Delta\sigma\,(r_{LL} - r_{LH})$. From (IC2), $r_{LH} > r_{LL}$. From the proof of Solution 1, $q_{LH} > q_{HL}$ for S' not very concave. Hence, (IC6) is slack. (IC7) and (IC8) are binding ($\gamma_1 > 0$ and $\gamma_2 > 0$) and thus satisfied. Using binding (IC1) in (IC8) and then binding (IC8) in (IC9), the latter becomes $r_{LL} \geq r_{HL}$, which is satisfied (as from proof of Solution 1). Using binding (IC1) in (IC8), then binding (IC8) in (IC10) and then binding (IC11) in (IC10), the latter is rewritten as $\Delta\theta\,(q_{LH} - q_{HH}) \geq \Delta\sigma\,(r_{LL} - r_{HL})$. Then, from (55), $\underline{q}_{LH} > \underline{q}_{HH}$ and $\overline{q}_{LH} > \overline{q}_{HH}$, so that $q_{LH} > q_{HH}$. Hence (IC10) is satisfied if $\Delta\theta\,(q_{HH} - q_{HH}) \geq \Delta\sigma\,(r_{LL} - r_{HL})$, which holds with equality ($\gamma_1 > 0$ and $\gamma_2 > 0$). (IC11) is binding (as from Lemma 3). Using the binding constraints (IC11) and (IC1) in (IC12), the latter is rewritten as $\Delta\theta\,(q_{LL} - q_{HL}) \geq \Delta\sigma\,(r_{LL} - r_{HL})$. Using (27), (IC12) is rewritten as $q_{LL} \geq q_{HH}$. In the proof of (IC4) here above, it was shown that this is strictly satisfied. Hence, (IC12) is slack.

D.2 Proof of Lemma 8

Take any given convexity of S', and first suppose that $\Delta\theta/\Delta\sigma$ is very close to 1. Then, rewriting (15) as (16) and recalling that, under (14), $\overline{q}_{HL} > \overline{q}_{HH}$, we see that (15) is satisfied.

Suppose now that $\gamma_1 = 1$. With $\Delta\theta/\Delta\sigma$ close to 1, this involves that $q_{HH} - q_{HL} > r_{LL} - r_{HL}$. Recalling that, for S' less convex, $q_{HH} - q_{HL} = r_{LL} - r_{HL}$ ((27) satisfied), and using (46) and (50), the shift from Solution 5, at which $\gamma_1 < 1$, to a solution at which $\gamma_1 = 1$ (hence, $\gamma_2 = 0$), is possible only if $\underline{q}_{HH} > \underline{q}_{LL}$ or, equivalently,

$$\frac{\Delta\theta}{\Delta\sigma} < (1-\nu)\,\frac{\mu + \gamma_2\,(1-\mu)}{(1-\gamma_2\nu)\,\mu}. \tag{59}$$

However, replacing $\gamma_2 = 0$ into (59), this comes out to be violated. The hypothesis that $\gamma_1 = 1$ leads to a contradiction. Next suppose that $\gamma_2 = 1$, meaning that $r_{LL} - r_{HL} > q_{HH} - q_{HL}$. With the right-hand side of (46) larger than that of (50) *i.e.*,

$$\frac{\Delta\theta}{\Delta\sigma} > \frac{\gamma_1}{\gamma_2}\frac{1-\mu}{\nu}, \tag{60}$$

and with (60) satisfied for $\gamma_2 = 1$ and $\gamma_1 = 0$, one can move from Solution 5 to a solution at which $\gamma_2 = 1$ only if $\underline{q}_{LL} > \underline{q}_{HH}$. That is, (59) must be violated *i.e.*, with $\gamma_2 = 0$, $\frac{\Delta\theta}{\Delta\sigma} > \frac{1-\nu}{(1-\nu)\mu}$. However, because $\frac{1-\nu}{(1-\nu)\mu} > 1$ whereas $\Delta\theta/\Delta\sigma$ is very close to 1, the hypothesis that $\gamma_2 = 1$ is contradicted. All in all, when $\Delta\theta/\Delta\sigma$ is close to 1 and S' is strictly convex, $\gamma_1 < 1$ and $\gamma_2 < 1$ at the solution to Γ.

Suppose now that $\Delta\theta/\Delta\sigma$ raises. Then, based on (46) and (50), we see that, for (27) to hold, it is necessary that γ_1 increases and γ_2 decreases.

We hereafter show that (15) tightens as $\Delta\theta/\Delta\sigma$ raises. With (27) satisfied, for (15) to hold it is necessary and sufficient that $r_{HH} - r_{LL} > 0$. Using the quantity solution in Lemma 4, we compute

$$S'(\overline{q}_{HH}) - S'(\overline{q}_{LL}) = \left(1 + \gamma_1 \frac{\nu}{1-\nu}\right)\Delta\theta + \left(1 + \gamma_2 \frac{1-\mu}{\mu}\right)\Delta\sigma, \tag{61}$$

from which $\overline{q}_{LL} > \overline{q}_{HH}$. We prove below that $\underline{q}_{LL} > \underline{q}_{HH}$ as well. We further compute

$$\begin{aligned} & [S'(\underline{q}_{HH}) - S'(\underline{q}_{LL})] - [S'(\overline{q}_{HH}) - S'(\overline{q}_{LL})] \\ = \; & -2\left[1 + \frac{(1-\beta)\nu\mu}{(1-\nu)(1-\mu)} + \gamma_2\left(\frac{(1-\beta)\nu}{1-\nu} + \frac{1-\mu}{\mu}\right)\right]\Delta\sigma, \end{aligned} \tag{62}$$

which for $\beta = 1$ becomes

$$[S'(\underline{q}_{HH}) - S'(\underline{q}_{LL})] - [S'(\overline{q}_{HH}) - S'(\overline{q}_{LL})] = -2\left(1 + \gamma_2 \frac{1-\mu}{\mu}\right)\Delta\sigma. \tag{63}$$

All else equal, the difference $r_{HH} - r_{LL}$ is smaller the smaller $\Delta\sigma$ and/or γ_2. Hence, as $\Delta\theta/\Delta\sigma$ raises and so γ_2 becomes smaller (as previously found), (15) is tightened.

We are left with checking whether it is possible to have a solution at which $\gamma_1 = 1$ in Case 2. Consider S' linear. With (27) satisfied, $r_{LL} - r_{HL} > q_{HH} - q_{HL}$. From (46) and (50), this is equivalent to (60) being violated. As S' becomes slightly convex, $r_{LL} - r_{HL}$ increases by some Δr and $q_{HH} - q_{HL}$ by some Δq. (27) holds for a range of small degrees of convexity of S'. As the convexity increases within that range, the ratio $\Delta r/\Delta q$ must remain constant and equal to $\Delta\theta/\Delta\sigma$. Take (27) to hold (hence, the degree of convexity to belong to the range aforementioned) and consider the ratio $\Delta r_1/\Delta q_1$ for some (slight) convexity of S', and the ratio $\Delta r_2/\Delta q_2$ for some more important convexity of S'. With $\underline{q}_{LL} > \underline{q}_{HH}$, if γ_1 and γ_2 remain unchanged, then $(\Delta r_1/\Delta q_1) < (\Delta r_2/\Delta q_2)$. Because it must be $(\Delta r_1/\Delta q_1) = (\Delta r_2/\Delta q_2)$, it is necessary that, when moving from less to more pronounced convexity, γ_1 and γ_2 change in a way such that $\Delta r/\Delta q$ takes a lower value than it would if γ_1 and γ_2 were unchanged. From (46) and (50), γ_1 must take a lower value and γ_2 a higher value so that γ_1 cannot jump to 1.

E Case 3

E.1 Proof of Solution 6

Start from the lowest degree of convexity of S' for which (15) is violated with $\beta = 1$. Then, at the solution, $\Pi_{LL,1} = \Pi_{LL,2}$. Equivalently,

$$(\Delta\theta + \Delta\sigma)(\overline{q}_{HL} - \overline{q}_{HH}) = (\Delta\theta - \Delta\sigma)(\underline{q}_{HH} - \underline{q}_{HL}). \tag{64}$$

From the proof of Solution 5, (15) is violated for some values of γ_1 and γ_2 for which (27) is satisfied. Then, using (45), we see that, as S' becomes more convex, (64) is still satisfied if β lowers. As γ_2 is to lower, in turn, for (27) to remain satisfied as S' becomes more convex, γ_2 and β decrease such that $r_{HH} = r_{LL}$, meaning that (64) and (27) are satisfied. At the limit, for some sufficiently high degree of convexity of S', $\beta = 0$ and we move to Solution 7.

Binding incentive constraints are the same as in Solution 5, except that (IC3) is here binding as well. Hence, one can prove that all remaining incentive constraints are satisfied in a similar manner.

E.2 Proof of Solution 7

Start from Solution 6 and suppose that S' is sufficiently convex, meaning that (64) is not satisfied for all β in $(0,1]$. (IC3) is more stringent than (IC1) so that $\beta = 0$. Moreover, as (15) is violated with strict inequality, $q_{HH} > q_{HL}$ and so $\gamma_3 = 0$. From the proof of Solution 6, recall that, at the solution, β falls to zero when (64) is not satisfied for all $\beta > 0$ for which (27) holds instead. This means that, when $\beta = 0$, $r_{LL} > r_{HH}$, which implies that $\gamma_2 = 1$.

E.2.1 Check the incentive constraints in Γ

(IC1) is slack and (IC3) is binding (as $\beta = 0$), hence they are both satisfied. As $\gamma_2 = 1$ and so (IC8) is binding, (IC2) is rewritten as $r_{LH} \geq r_{LL}$ and is satisfied (recall (52) and (56)). Further using (IC8) in (IC4), the latter is rewritten as $q_{LL} \geq q_{HL}$. From Lemma 4, for $\beta = 0$ and $\gamma_2 = 1$ we compute

$$\begin{aligned} S'(\underline{q}_{HL}) - S'(\underline{q}_{LL}) &= \Delta\theta + \frac{\nu}{(1-\nu)\,\mu}\Delta\sigma \\ S'(\overline{q}_{HL}) - S'(\overline{q}_{LL}) &= \Delta\theta - \frac{\nu}{(1-\nu)\,\mu}\Delta\sigma \end{aligned}$$

Hence, $\underline{q}_{LL} > \underline{q}_{HL}$. If $\overline{q}_{LL} > \overline{q}_{HL}$, then (IC4) is slack. If $\overline{q}_{LL} < \overline{q}_{HL}$, then we compute

$$\left[S'(\underline{q}_{HL}) - S'(\underline{q}_{LL})\right] - \left[S'(\overline{q}_{LL}) - S'(\overline{q}_{HL})\right] = 2\Delta\theta$$

and we deduce that $q_{LL} > q_{HL}$ as long as S' is convex, and so (IC4) is slack. As (IC11) is binding (Lemma 3), (IC5) becomes $r_{HH} \geq r_{HL}$. Under (14), it follows from Lemma 5 that $\underline{q}_{HH} > \underline{q}_{HL}$ and $\overline{q}_{HL} > \overline{q}_{HH}$ so that (IC5) is slack. Using (IC8) and (IC1) in (IC6), the latter is satisfied if and only if $\Delta\theta\,(q_{LH} - q_{HL}) \geq \Delta\sigma\,(r_{LL} - r_{LH})$. This holds because $r_{LH} > r_{LL}$ (from (IC2)) and $q_{LH} > q_{HL}$ (from the proof of Solution 1). (IC7) is slack $(\gamma_1 = 0)$, (IC8) is binding $(\gamma_2 = 1)$ and (IC9) is slack $(\gamma_3 = 0)$. Using (IC3) in (IC8), and then (IC8) in (IC10), the latter is written as $\Delta\theta\,(q_{LH} - q_{HH}) \geq \Delta\sigma\,(r_{LL} - r_{HH})$. This is equivalent to

$$(\Delta\theta - \Delta\sigma)\,(\underline{q}_{LH} - \underline{q}_{HH}) + (\Delta\sigma + \Delta\theta)\,(\overline{q}_{LH} - \overline{q}_{HH}) + \Delta\sigma(\overline{q}_{LL} - \overline{q}_{LH}) + \Delta\sigma(\underline{q}_{LH} - \underline{q}_{LL}) \geq 0. \quad (65)$$

Recall that, under (52) and (56), $\overline{q}_{LL} > \overline{q}_{LH}$ and $\underline{q}_{LH} > \underline{q}_{LL}$. Furthermore, under Lemma 4, for $\beta = 0$ and $\gamma_2 = 1$ we compute

$$\begin{aligned} S'(\underline{q}_{HH}) - S'(\underline{q}_{LH}) &= \Delta\theta + \frac{\nu}{(1-\nu)\,(1-\mu)}\,(\Delta\theta - \Delta\sigma) \\ S'(\overline{q}_{HH}) - S'(\overline{q}_{LH}) &= \Delta\theta + \frac{\nu}{(1-\nu)\,(1-\mu)}\,(\Delta\theta + \Delta\sigma), \end{aligned}$$

so that $\underline{q}_{LH} > \underline{q}_{HH}$ and $\overline{q}_{LH} > \overline{q}_{HH}$. It follows that the left-hand side of (IC10) is positive and so (IC10) is slack. (IC11) is binding (from Lemma 3). Using (IC3) in (IC12), the latter is rewritten as $\Delta\theta\,(q_{LL} - q_{HH}) \geq \Delta\sigma\,(r_{LL} - r_{HH})$. Using (IC10), for (IC12) to hold it suffices that $q_{LH} \geq q_{LL}$. Recalling from (56) and (52) that

$$S'(\underline{q}_{LL}) - S'(\underline{q}_{LH}) = S'(\overline{q}_{LH}) - S'(\overline{q}_{LL}),$$

it follows that $q_{LH} \geq q_{LL}$ for S' convex. Hence, (IC12) is satisfied.

F Case 4

Recall from Lemma (5) that, when (14) is violated, $\underline{q}_{HH} = \underline{q}_{HL}$ and $\overline{q}_{HL} = \overline{q}_{HH}$. The rents in Lemma 3 are rewritten as

$$\begin{aligned}\Pi_{HL} &= 0;\ \Pi_{HH} = \Delta\sigma r_{HH};\ \Pi_{LL} = \Delta\theta q_{HH} \\ \Pi_{LH} &= \gamma_1 \left(\Delta\theta q_{HH} + \Delta\sigma r_{HH}\right) + \gamma_2 \left(\Delta\theta q_{HH} + \Delta\sigma r_{LL}\right).\end{aligned} \tag{66}$$

The agent's expected rent is then

$$\nu\Delta\theta q_{HH} + (1-\mu)\left\{\left[1-(1-\gamma_1)\nu\right] r_{HH} + \nu\gamma_2 r_{LL}\right\}\Delta\sigma. \tag{67}$$

Denote Γ" the reduced problem, similar to Γ′, in which the expected rent is yet given by (67). Then, at optimum, the quantities assigned to type LH and type LL are still pinned down as in Lemma 4. Instead, the quantities assigned to type HH and HL are characterized as

$$\begin{aligned}S'(\underline{q}_{HH}) &= \theta_H - \sigma_H + \frac{\nu}{(1-\nu)(1-\mu)}\Delta\theta + \frac{1-(1-\gamma_1)\nu}{1-\nu}\Delta\sigma \\ S'\left(\overline{q}_{HH}\right) &= \theta_H + \sigma_H + \frac{\nu}{(1-\nu)(1-\mu)}\Delta\theta - \frac{1-(1-\gamma_1)\nu}{1-\nu}\Delta\sigma,\end{aligned}$$

together with $\underline{q}_{HH} = \underline{q}_{HL}$ and with $\overline{q}_{HL} = \overline{q}_{HH}$. In what follows, we analyze separately the three possible situations, namely (1) $\gamma_1 = 1$ ($r_{HH} > r_{LL}$ in (66)), (2) $\gamma_1 > 0$ and $\gamma_2 > 0$ ($r_{HH} = r_{LL}$), and (3) $\gamma_2 = 1$ ($r_{LL} > r_{HH}$), identifying the conditions under which they arise and checking that the solutions satisfy the incentive constraints in Γ.

F.1 The case of $\gamma_1 = 1$ (Solution 8)

We check first under which condition $\gamma_1 = 1$. From (66), this means that $r_{HH} > r_{LL}$. We have

$$S'(\underline{q}_{HH}) - S'(\underline{q}_{LL}) = \frac{1-\mu(1-\nu)}{(1-\nu)(1-\mu)}\Delta\theta + \frac{\nu}{1-\nu}\Delta\sigma$$

so that $\underline{q}_{LL} > \underline{q}_{HH}$. Moreover,

$$S'\left(\overline{q}_{HH}\right) - S'\left(\overline{q}_{LL}\right) = \frac{1-\mu(1-\nu)}{(1-\nu)(1-\mu)}\Delta\theta - \frac{\nu}{1-\nu}\Delta\sigma,$$

which is positive and so $\overline{q}_{LL} > \overline{q}_{HH}$.Computing

$$\left[S'(\underline{q}_{HH}) - S'(\underline{q}_{LL})\right] - \left[S'\left(\overline{q}_{HH}\right) - S'\left(\overline{q}_{LL}\right)\right] = 2\frac{\nu}{1-\nu}\Delta\sigma \tag{68}$$

we deduce that $r_{HH} > r_{LL}$ if and only if S' is sufficiently concave.

(IC1) and (IC3) are binding. We bind (IC11) and (IC7); we use the former in the latter, and then (IC7) in (IC2). Then, (IC2) is rewritten as $r_{LH} \geq r_{HH}$. We compute

$$S'(\underline{q}_{HH}) - S'(\underline{q}_{LH}) = \frac{1-\mu(1-\nu)}{(1-\nu)(1-\mu)}\Delta\theta + \frac{1}{1-\nu}\Delta\sigma$$

so that $\underline{q}_{LH} > \underline{q}_{HH}$. Moreover,

$$S'(\overline{q}_{HH}) - S'(\overline{q}_{LH}) = \frac{1-\mu(1-\nu)}{(1-\nu)(1-\mu)}\Delta\theta - \frac{1}{1-\nu}\Delta\sigma$$

so that $\overline{q}_{LH} > \overline{q}_{HH}$ if and only if $\frac{\Delta\theta}{\Delta\sigma} > \frac{1-\mu}{1-\mu+\nu\mu}$, which is actually the case. We further compute

$$[S'(\underline{q}_{HH}) - S'(\underline{q}_{LH})] - [S'(\overline{q}_{HH}) - S'(\overline{q}_{LH})] = \frac{2}{1-\nu}\Delta\sigma, \tag{69}$$

which shows that, at the solution to Γ'', $r_{LH} \geq r_{HH}$ (*i.e.*, (IC2) holds) if and only if S' is not very concave. Using (IC1) in (IC4), together with the fact that quantities of types HH and HL are pooled, (IC4) is rewritten as $q_{LL} \geq q_{HH}$ and it is slack provided $\underline{q}_{LL} > \underline{q}_{HH}$ and $\overline{q}_{LL} > \overline{q}_{HH}$ (as found above). Using (IC11) in (IC5), the latter is rewritten as $r_{HH} \geq r_{HL}$ and it is satisfied with equality as the quantities of types HH and HL are pooled. Using (IC11) in (IC7), and then (IC7) in (IC6), the latter is rewritten as $\Delta\theta q_{LH} + \Delta\sigma r_{LH} \geq \Delta\theta q_{HH} + \Delta\sigma r_{HH}$, and it is implied by (IC2) and (IC10) (as written here below). (IC7) is binding $(\gamma_1 = 1)$, (IC8) is slack $(\gamma_2 = 0)$. (IC9) collapses onto (IC7) as the quantities of types HH and HL are pooled. Using (IC7) in (IC10), the latter is rewritten as $q_{LH} \geq q_{HH}$. We found that $\underline{q}_{LH} > \underline{q}_{HH}$ and $\overline{q}_{LH} > \overline{q}_{HH}$, hence (IC10) is slack. (IC11) was taken to be binding. (IC12) is rewritten as $\Delta\theta q_{LL} + \Delta\sigma r_{HH} \geq \Delta\theta q_{HH} + \Delta\sigma r_{LL}$. This is implied by (IC4) together with $r_{HH} > r_{LL}$, which was taken to hold.

F.1.1 When the solution to Γ'' violates incentive constraints in Γ

The sole conditions that might not hold jointly in Γ'' are $r_{HH} > r_{LL}$ and $r_{LH} \geq r_{HH}$ ((IC2)). As proved above, $r_{HH} > r_{LL}$ holds but the condition $r_{LH} \geq r_{HH}$ does not when S' is very concave. In this case, the solution to Γ is such that (IC2) is binding, or, equivalently, $r_{LH} = r_{HH}$. Moreover, by comparing (68) with (69), we notice that there exist some intermediate degrees of concavity of S' for which both $r_{HH} > r_{LL}$ and $r_{LH} > r_{HH}$. The new reduced programme is then that in which P faces the same maximization problem as above, but subject to the constraint $r_{LH} = r_{HH}$. Production levels are pinned down as in Lemma 4 for type LL. For types HH, HL and LH, they are characterized by attaching the Lagrange multiplier $\lambda_3 > 0$ to the new constraint $r_{LH} = r_{HH}$.

F.2 The case of $\gamma_2 = 1$ (Solution 10)

We check first under which condition $\gamma_2 = 1$. From (66), this means that $r_{LL} > r_{HH}$. We have

$$S'(\overline{q}_{HH}) - S'(\overline{q}_{LL}) = \frac{1-\mu(1-\nu)}{(1-\nu)(1-\mu)}\Delta\theta + \frac{1-\mu}{\mu}\Delta\sigma$$

and see that $\overline{q}_{LL} > \overline{q}_{HH}$. We further compute

$$S'(\underline{q}_{HH}) - S'(\underline{q}_{LL}) = \frac{1-\mu(1-\nu)}{(1-\nu)(1-\mu)}\Delta\theta - \frac{1-\mu}{\mu}\Delta\sigma$$

so that $\underline{q}_{LL} > \underline{q}_{HH}$ if and only if

$$\frac{\Delta\theta}{\Delta\sigma} \geq \frac{(1-\nu)(1-\mu)^2}{\mu[1-\mu(1-\nu)]}. \tag{70}$$

This is verified if the right-hand side of (70) is smaller than the right-hand side of (14) evaluated at $\beta = 0$ and $\gamma_2 = 1$ *i.e.*,

$$\frac{1-\mu}{\frac{1}{1-\nu}-\mu} < \frac{1}{\nu} + \frac{\mu}{1-\mu}.$$

This inequality is satisfied because the left-hand side is smaller than 1, whereas the right-hand side is larger than 1. Hence, $\underline{q}_{LL} > \underline{q}_{HH}$. Using the calculations above, we further compute

$$[S'(\overline{q}_{HH}) - S'(\overline{q}_{LL})] - \left[S'(\underline{q}_{HH}) - S'(\underline{q}_{LL})\right] = 2\frac{1-\mu}{\mu}\Delta\sigma$$

from which we deduce that $r_{LL} > r_{HH}$ if and only if S' is sufficiently convex.

(IC1) and (IC3) are both binding. Using the binding constraint (IC8) in (IC2), the latter is rewritten as $r_{LH} \geq r_{LL}$. We compute

$$S'(\underline{q}_{LL}) - S'(\underline{q}_{LH}) = S'(\overline{q}_{LH}) - S'(\overline{q}_{LL}) = \frac{\Delta\sigma}{\mu}$$

and see that $\underline{q}_{LH} > \underline{q}_{LL}$ and $\overline{q}_{LL} > \overline{q}_{LH}$ so that (IC2) is satisfied. Using (IC1) in (IC4) and provided the quantities of types HH and HL are pooled, (IC4) is rewritten as $q_{LL} \geq q_{HH}$ and it is slack as we found that $\overline{q}_{LL} > \overline{q}_{HH}$ and $\underline{q}_{LL} > \underline{q}_{HH}$. Using the binding constraint (IC11) in (IC5), the latter is rewritten as $r_{HH} \geq r_{HL}$, which is satisfied as an equality, provided the quantities of types HH and HL are pooled in Case 4. Using (IC1) in (IC8), and then (IC8) in (IC6), the latter is rewritten as $\Delta\theta\,(q_{LH} - q_{HH}) \geq \Delta\sigma\,(r_{LL} - r_{LH})$. This is satisfied because $r_{LH} \geq r_{LL}$ (from (IC2)) and $q_{LH} \geq q_{HH}$. To see that the latter inequality is satisfied, observe that it is necessary for (IC10) to hold (with $r_{LL} > r_{HH}$, by assumption), which is shown to occur below. (IC7) is slack ($\gamma_1 = 0$) and (IC8) binding ($\gamma_2 = 1$). (IC9) is equivalent to (IC7) as the quantities of types HH and HL are pooled. Using (IC3) in (IC8), and then (IC8) in (IC10), the latter is rewritten as $\Delta\theta\,(q_{LH} - q_{HH}) \geq \Delta\sigma\,(r_{LL} - r_{HH})$. This is equivalent to (65) and observing that, as before, its left-hand side is positive, it comes out that (IC10) is satisfied when $r_{LL} > r_{HH}$. Lastly, (IC12) is rewritten as $\Delta\theta\,(q_{LL} - q_{HH}) \geq \Delta\sigma\,(r_{LL} - r_{HH})$ or, equivalently,

$$(\Delta\theta + \Delta\sigma)\,(\overline{q}_{LL} - \overline{q}_{HH}) \geq (\Delta\theta - \Delta\sigma)\,(\underline{q}_{HH} - \underline{q}_{LL}).$$

As $\overline{q}_{LL} > \overline{q}_{HH}$ and $\underline{q}_{LL} > \underline{q}_{HH}$, this condition holds as well.

F.3 The case of $\gamma_1 > 0$ and $\gamma_2 > 0$ (Solution 9)

We found that $r_{HH} > r_{LL}$ if and only if S' is sufficiently concave, whereas $r_{LL} > r_{HH}$ if and only if S' is sufficiently convex. Therefore, when S' is almost linear, it must be the case that $r_{LL} = r_{HH}$, meaning that both $\gamma_1 > 0$ and $\gamma_2 > 0$. The incentive constraints in Γ are verified following the same procedure as above.

G *Ex-post* participation constraints

Suppose that $(\overline{pc}_{HL})$ is binding and that $(\overline{pc}_{ij})$ is slack for all $ij \neq HL$. Then, $\Pi_{HL} = \sigma_L \overline{q}_{HL}$ at optimum. All other rents are the same as in Γ plus the additional term $\sigma_L \overline{q}_{HL}$. Following the same reasoning as for Γ, if $\overline{q}_{HL} > \overline{q}_{HH}$ and $\underline{q}_{HH} > \underline{q}_{HL}$ at optimum, and if all upward incentive constraints are slack, quantities are characterized as in Lemma 4, except that:

$$S'(\overline{q}_{HL}) = \theta_H + 2\sigma_L + \frac{\nu}{1-\nu}\left\{\left[\beta + (\gamma_2\beta + \gamma_3)\frac{1-\mu}{\mu}\right]\Delta\theta - \left[1 - \beta + (1-\nu\gamma_2\beta)\frac{1-\mu}{\nu\mu}\right]\Delta\sigma\right\}.$$

In this setting, (45) becomes

$$[S'(\underline{q}_{HL}) - S'(\underline{q}_{HH})] - [S'(\overline{q}_{HH}) - S'(\overline{q}_{HL})] = 2\left(\beta\gamma_2 + \gamma_3 - \frac{(1-\beta)\mu}{1-\mu}\right)\frac{\nu}{(1-\nu)\mu}\Delta\theta + \sigma_L.$$

If $\beta = 1$, then $q_{HH} > q_{HL}$ if and only if S' is not sufficiently concave. That is, (15) holds for S' sufficiently concave and for S' almost linear, not otherwise. Hence, in the particular case in which S' is almost linear, $\beta = 1$ and so $\Pi_{LL,1} > \Pi_{LL,2}$. In turn, (50) becomes

$$[S'(\underline{q}_{HL}) - S'(\underline{q}_{LL})] - [S'(\overline{q}_{HL}) - S'(\overline{q}_{LL})] = 2\frac{(\gamma_1+\gamma_3)(1-\mu)}{(1-\nu)\mu}\Delta\sigma - \sigma_L. \tag{71}$$

In Γ, the right-hand side of (50) is positive. Here, the right-hand side of (71) is positive if and only if

$$\sigma_L < 2\frac{(\gamma_1+\gamma_3)(1-\mu)}{(1-\nu)\mu}\Delta\sigma. \tag{72}$$

Take (72) to hold. Then, as in Γ, $r_{LL} > r_{HL}$ if and only if S' is not sufficiently concave. Overall, for S' almost linear and $\gamma_2 + \gamma_3 > 0$, $q_{HH} > q_{HL}$ and $r_{LL} > r_{HL}$, meaning that $\gamma_3 = 0$. Suppose $\gamma_2 = 1$. Then, $q_{HH} > q_{HL}$ and $r_{LL} < r_{HL}$, which yields a contradiction. Hence, two cases are feasible *i.e.*, $\gamma_1 > 0$ together with $\gamma_2 > 0$, and $\gamma_1 = 1$. For $\Delta\theta/\Delta\sigma$ sufficiently small, the former case arises.

Let us check the conditions that we took to hold. We took $\underline{q}_{HH} > \underline{q}_{HL}$, which occurs indeed when $\beta = 1$. Then, $\overline{q}_{HL} > \overline{q}_{HH}$ if and only if

$$\frac{\Delta\theta}{\Delta\sigma} < \frac{1}{1-\gamma_1}\left[\frac{1}{\nu} - (1-\mu)\left(\gamma_2 + \frac{\mu}{1-\mu}\right) - \frac{\mu(1-\nu)}{\nu}\frac{\sigma_L}{\Delta\sigma}\right].$$

Condition (72) becomes

$$\sigma_L < 2\frac{\gamma_1(1-\mu)}{(1-\nu)\mu}\Delta\sigma. \tag{73}$$

Furthermore, $\underline{q}_{HH} > \overline{q}_{HH}$ if and only if

$$\sigma_L > \frac{2}{3}\left(1 + \gamma_1\frac{\nu}{1-\nu}\right)\frac{1-\mu}{\mu}\Delta\sigma. \tag{74}$$

For (73) and (74) to hold together, it is necessary that

$$\sigma_L \in \left(\frac{2}{3}\left(1 + \gamma_1\frac{\nu}{1-\nu}\right)\frac{1-\mu}{\mu}\Delta\sigma, 2\frac{\gamma_1(1-\mu)}{(1-\nu)\mu}\Delta\sigma\right),$$

which further requires that γ_1 be large enough *i.e.*, $\gamma_1 \geq (1-\nu)/(3-\nu)$.

Lastly, for $(\overline{pc}_{ij})$ to be slack for all $ij \neq HL$, it must be the case that, at the solution:

$$\sigma_L \overline{q}_{HL} + \Delta\sigma r_{HL} \geq \sigma_H \overline{q}_{HH} \tag{75}$$
$$\sigma_L \overline{q}_{HL} + \Delta\theta q_{HL} \geq \sigma_L \overline{q}_{LL} \tag{76}$$
$$\Delta\theta q_{HH} + \Delta\sigma r_{HL} \geq \sigma_H \overline{q}_{LH}. \tag{77}$$

G.1 An application to the award of for monopoly franchises

P runs an auction to award the contract for production of the good. The auction is competitive. For simplicity, there are two participants, denoted 1 and 2. Each observes privately his own type $i_k j_k \in \Upsilon$, $k \in \{1,2\}$. Define $Q^{i_{-k}j_{-k}}_{i'_k j'_k}$ the probability that participant k wins the auction with the announcement $i'_k j'_k$, if the other participant has type $i_{-k}j_{-k}$. $\gamma_3 = 0$ if and only if

$$\begin{aligned} E_{i_{-k}j_{-k}}\left(Q^{i_{-k}j_{-k}}_{LH}\Pi_{LH,4}\right) &> E_{i_{-k}j_{-k}}\left(Q^{i_{-k}j_{-k}}_{LH}\Pi_{LH,1}\right) \\ E_{i_{-k}j_{-k}}\left(Q^{i_{-k}j_{-k}}_{LH}\Pi_{LH,4}\right) &> E_{i_{-k}j_{-k}}\left(Q^{i_{-k}j_{-k}}_{LH}\Pi_{LH,2}\right), \end{aligned}$$

where Π_{ij} have the same formulas as in problem Γ_{ep}. These inequalities are, same as before, equivalent to

$$\Pi_{LH,4} > \Pi_{LH,1};\ \Pi_{LH,4} > \Pi_{LH,2}$$

We thus need to check the same conditions as in Γ and Γ_{ep}. In particular, to have $\gamma_3 = 0$, it is necessary that either $q_{HH} - q_{HL} > 0$ or $r_{LL} - r_{HL} > 0$, or both.

Letting Q_{ij} denote the probability that at least one participant has type ij and wins the auction, the expected return of P is given by

$$\begin{aligned} &\sum_{ij\in\Gamma} \frac{1}{2} Q_{ij} \left(S(\underline{q}_{ij}) - (\theta_i - \sigma_j)\, \underline{q}_{ij} + S(\overline{q}_{ij}) - (\theta_i + \sigma_j)\, \overline{q}_{ij} \right) - Q_{LL}\left[\beta \Pi_{LL,1} + (1-\beta)\,\Pi_{LL,2}\right] \\ &- Q_{HH}\Pi_{HH} - Q_{LH}\left\{\gamma_1 \Pi_{LH,1} + \gamma_2\left[\beta\Pi_{LH,2} + (1-\beta)\,\Pi_{LH,3}\right] + \gamma_3 \Pi_{LH,4}\right\} - \sigma_L \overline{q}_{HL}, \end{aligned}$$

where

$$\begin{aligned} Q_{LL} &= \nu\mu\left[\nu\mu + 2(1-\nu)\right] \\ Q_{HH} &= (1-\nu)^2\left(1-\mu^2\right) \\ Q_{LH} &= \nu(1-\mu)\left[2 - \nu(1-\mu)\right] \\ Q_{HL} &= (1-\nu)^2\mu^2. \end{aligned}$$

Type LH is assigned the FB quantities. The optimal LL-quantities are such that

$$\begin{aligned} S'(\underline{q}_{LL}) &= \theta_L - \sigma_L + \gamma_2 \frac{(1-\mu)\left[2-\nu(1-\mu)\right]}{\mu\left[2(1-\nu)+\nu\mu\right]}\Delta\sigma \\ S'(\overline{q}_{LL}) &= \theta_L + \sigma_L - \gamma_2 \frac{(1-\mu)\left[2-\nu(1-\mu)\right]}{\mu\left[2(1-\nu)+\nu\mu\right]}\Delta\sigma. \end{aligned}$$

The optimal HH-quantities are such that

$$\begin{aligned}
S'(\underline{q}_{HH}) &= \theta_H - \sigma_H + \frac{\nu\left\{\left[(1-\beta)\mu + \left[\gamma_1 + (1-\beta)\gamma_2\right](1-\mu)\right]\left[2-\nu(1-\mu)\right] - (1-\beta)\nu\mu\right\}}{(1-\nu)^2(1-\mu^2)}\Delta\theta \\
&\quad - \frac{(1-\beta)\nu\left\{\left[\mu+\gamma_2(1-\mu)\right]\left[2-\nu(1-\mu)\right]-\nu\mu\right\}}{(1-\nu)^2(1-\mu^2)}\Delta\sigma \\
S'(\overline{q}_{HH}) &= \theta_H + \sigma_H + \frac{\nu\left\{\left[(1-\beta)\mu + \left[\gamma_1 + (1-\beta)\gamma_2\right](1-\mu)\right]\left[2-\nu(1-\mu)\right] - (1-\beta)\nu\mu\right\}}{(1-\nu)^2(1-\mu^2)}\Delta\theta \\
&\quad + \frac{(1-\beta)\nu\left\{\left[\mu+\gamma_2(1-\mu)\right]\left[2-\nu(1-\mu)\right]-\nu\mu\right\}}{(1-\nu)^2(1-\mu^2)}\Delta\sigma.
\end{aligned}$$

The optimal HL-quantities are such that

$$\begin{aligned}
S'(\underline{q}_{HL}) &= \theta_H - \sigma_L + \frac{\nu\left\{\left[\beta\mu + (1-\mu)(\gamma_2\beta+\gamma_3)\right]\left[2-\nu(1-\mu)\right]-\beta\nu\mu\right\}}{(1-\nu)^2\mu^2}\Delta\theta \\
&\quad + \frac{\nu}{(1-\nu)^2\mu^2}\left\{\left[1-\beta\gamma_2(1-\mu)-\beta\mu\right]\left[2-\nu(1-\mu)\right]\right. \\
&\quad \left. + \frac{(1-\nu)^2(1-\mu^2)}{\nu} - (1-\beta)\nu\mu\right\}\Delta\sigma \\
S'(\overline{q}_{HL}) &= \theta_H + \sigma_L + \frac{\nu\left\{\left[\beta\mu + (\gamma_2\beta+\gamma_3)(1-\mu)\right]\left[2-\nu(1-\mu)\right]-\beta\nu\mu\right\}}{(1-\nu)^2\mu^2}\Delta\theta \\
&\quad - \frac{\nu}{(1-\nu)^2\mu^2}\left\{\left[1-\beta\gamma_2(1-\mu)-\beta\mu\right]\left[2-\nu(1-\mu)\right]\right. \\
&\quad \left. + \frac{(1-\nu)^2(1-\mu^2)}{\nu} - (1-\beta)\nu\mu\right\}\Delta\sigma + \sigma_L
\end{aligned}$$

Suppose that S' is almost linear and that $\Delta\theta/\Delta\sigma$ is small at the solution *i.e.*, the optimal quantities characterized above are such that $\overline{q}_{HL} > \overline{q}_{HH}$ (the counterpart for (14)). We hereafter check under which conditions the solution is such that $\beta = 1$, $\gamma_3 = 0$, $\gamma_1 > 0$, and $\gamma_2 > 0$, as we found in Γ and Γ_{ep}. We compute

$$\begin{aligned}
&\left[S'(\underline{q}_{HL}) - S'(\underline{q}_{HH})\right] - \left[S'(\overline{q}_{HH}) - S'(\overline{q}_{HL})\right] \qquad (78)\\
&= 2\nu\frac{\left[(\gamma_2\beta+\gamma_3)(1-\mu) + (\beta-\mu)\mu\right]\left[2-\nu(1-\mu)\right] - (\beta-\mu^2)\nu\mu}{(1-\nu)^2\mu^2(1-\mu^2)}\Delta\theta + \sigma_L.
\end{aligned}$$

When $\beta = 1$, the right-hand side of (78) is

$$2\nu\frac{(\gamma_2+\gamma_3+\mu)\left[2-\nu(1-\mu)\right]-\nu\mu(1+\mu)}{(1-\nu)^2\mu^2(1+\mu)}\Delta\theta + \sigma_L > 0.$$

Hence, $q_{HH} > q_{HL}$ if and only if S' is almost linear or convex, which confirms that $\gamma_3 = 0$. Moreover, we know from the proof of Solution 5 and Lemma 8 that with $\gamma_3 = 0$, $\gamma_1 > 0$ and $\gamma_2 > 0$, $\beta = 1$ if and only if $r_{HH} > r_{LL}$. Here, (63) becomes

$$\left[S'(\underline{q}_{HH}) - S'(\underline{q}_{LL})\right] - \left[S'(\overline{q}_{HH}) - S'(\overline{q}_{LL})\right] = -2\left(1 + \gamma_2\frac{(1-\mu)\left[2-\nu(1-\mu)\right]}{\mu\left[2(1-\nu)+\nu\mu\right]}\right)\Delta\sigma,$$

confirming that $r_{HH} > r_{LL}$ if and only if S' is not sufficiently convex. Hence, $\beta = 1$ for S' almost linear. Replacing $\beta = 1$ in (78) yields

$$[S'(\underline{q}_{HL}) - S'(\underline{q}_{HH})] - [S'(\overline{q}_{HH}) - S'(\overline{q}_{HL})] = \frac{2\nu}{(1-\nu)^2 \mu^2} \left(\gamma_2 \frac{2 - \nu(1-\mu)}{1+\mu} + 2\mu \frac{1-\nu}{1+\mu} \right) \Delta\theta + \sigma_L.$$

Moreover, $\beta = 1$ and $\gamma_3 = 0$ we compute

$$\begin{aligned} & [S'(\underline{q}_{HL}) - S'(\underline{q}_{LL})] - [S'(\overline{q}_{HL}) - S'(\overline{q}_{LL})] \\ = \; & \frac{2(1-\mu)}{\mu^2} \left[\gamma_1 \frac{\nu[2 - \nu(1-\mu)]}{(1-\nu)^2} + \mu^2 \frac{(1+\mu)(1-\nu)^2}{\nu} \right] \Delta\sigma - \sigma_L \end{aligned}$$

When $\gamma_1 > 0$ and $\gamma_2 > 0$, as $\gamma_1 + \gamma_2 = 1$, raising either the difference $q_{HH} - q_{HL}$ or the difference $r_{LL} - r_{HL}$ involves reducing the other. However, unlike in the proof of Solution 5, it is possible to have $q_{HH} > q_{HL}$ together with $r_{LL} > r_{HL}$ when either $\gamma_1 = 1$ or $\gamma_2 = 1$. Therefore, a solution can arise, at which either $\gamma_1 > 0$ and $\gamma_2 > 0$ (as before), but also $\gamma_1 = 1$ or $\gamma_2 = 1$.

H Examples

In all numerical examples, we used the surplus function

$$S(q) = aq - \frac{q^{1+b}}{1+b},$$

with $a > 0$, $b > 0$, such that $S(q) > 0$ for all the quantities that we found to solve Γ, and $S'(q) = a - q^b > 0$. Then, $S''(q) = -bq^{b-1} < 0$, and $S'''(q) = -b(b-1)q^{b-2}$ is positive for $b < 1$ and negative for $b > 1$. The quantities pinned down in Lemma 4 are computed as follows. For type HL :

$$\begin{aligned} \underline{q}_{HL} &= \left[a - \left(\theta_H - \sigma_L + \nu \frac{\mu + \gamma_2(1-\mu)}{\mu(1-\nu)} \Delta\theta + (1-\mu) \frac{1 - (1-\gamma_1)\nu}{\mu(1-\nu)} \Delta\sigma \right) \right]^{\frac{1}{b}} \\ \overline{q}_{HL} &= \left[a - \left(\theta_H + \sigma_L + \nu \frac{\mu + \gamma_2(1-\mu)}{\mu(1-\nu)} \Delta\theta - (1-\mu) \frac{1 - (1-\gamma_1)\nu}{\mu(1-\nu)} \Delta\sigma \right) \right]^{\frac{1}{b}}. \end{aligned}$$

For type HH :

$$\begin{aligned} \underline{q}_{HH} &= \left[a - \left(\theta_H - \sigma_H + \gamma_1 \frac{\nu}{1-\nu} \Delta\theta \right) \right]^{\frac{1}{b}} \\ \overline{q}_{HH} &= \left[a - \left(\theta_H + \sigma_H + \gamma_1 \frac{\nu}{1-\nu} \Delta\theta \right) \right]^{\frac{1}{b}}. \end{aligned}$$

For type LL :

$$\begin{aligned}\underline{q}_{LL} &= \left[a - \left(\theta_L - \sigma_L + \gamma_2 \frac{1-\mu}{\mu}\Delta\sigma\right)\right]^{\frac{1}{b}} \\ \overline{q}_{LL} &= \left[a - \left(\theta_L + \sigma_L - \gamma_2 \frac{1-\mu}{\mu}\Delta\sigma\right)\right]^{\frac{1}{b}}.\end{aligned}$$

For type LH :

$$\begin{aligned}\underline{q}_{LH} &= \left[a - (\theta_L - \sigma_H)\right]^{\frac{1}{b}} \\ \overline{q}_{LH} &= \left[a - (\theta_L + \sigma_H)\right]^{\frac{1}{b}}.\end{aligned}$$

H.1 Example 1

In the table hereafter, we report the values that the optimal quantities take for $b = 1$ and $b = 0.4$, at which γ_1 and γ_2 are such that (27) holds and $\gamma_1 = 1 - \gamma_2$:

b	$\underline{q}_{HL}$	$\overline{q}_{HL}$	$\underline{q}_{HH}$	$\overline{q}_{HH}$	$\underline{q}_{LL}$	$\overline{q}_{LL}$	$\underline{q}_{LH}$	$\overline{q}_{LH}$
1	11.45	7.35	12.95	6.35	11.45	8.12	14.3	7.7
0.5	133.05	56.3	166.25	39.61	195	65	205	59.29

H.2 Example 2

In the table hereafter, we report the values that the optimal quantities take, together with the necessary condition (14) :

σ_H	$\underline{q}_{HL}$	$\overline{q}_{HL}$	$\underline{q}_{HH}$	$\overline{q}_{HH}$	$\underline{q}_{LL}$	$\overline{q}_{LL}$	$\underline{q}_{LH}$	$\overline{q}_{LH}$	*Condition* (14)
3.9	90.6	73.35	187.65	34.8	166	83	222	50.41	$1.11 < 5$
3.6	109.02	64.82	177.17	37.34	183.47	71.49	213.16	54.76	$1.66 < 8.63$
3.3	133.05	56.3	166.25	39.61	195	65	205	59.29	$3.33 < 57.63$

H.3 Example 3

With $(\overline{pc}_{HL})$ binding, the HL-quantity in the bad state is pinned down as follows:

$$\overline{q}_{HL} = a - \left(\theta_H + 2\sigma_L + \nu \frac{\mu + \gamma_2 (1-\mu)}{\mu(1-\nu)}\Delta\theta - (1-\mu)\frac{1-(1-\gamma_1)\nu}{\mu(1-\nu)}\Delta\sigma\right).$$

All other quantitites are characterized as above for $b = 1$. At the solution, $\gamma_1 = 0.9$ and $\gamma_2 = 0.1$. The conditions that need to be satisfied are as follows:

$$\begin{aligned}(36) &: 1.5 < 13.4; \quad (15) : 4.2 < 5.98; \quad (73) : 3 < 8.25 \\ (75) &: 22.62 > 21; \quad (76) : 31.67 > 24.93; \quad (77) : 33.82 > 27.36\end{aligned}$$