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Abstract

This paper focuses on the existence of a Cournot equilibrium in a n-firm Cournot market for a single homogeneous commodity. Using a simple argument and proof, it shows that a Cournot equilibrium exists if each firm's marginal revenue declines with its own output and some weak non-decreasing incremental cost condition is satisfied.

Keywords: Cournot Competition; existence of Cournot equilibrium; supermodular games.

JEL Codes: L13, C72, C62

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1 Introduction

In industrial organization, the Cournot model is an essential tool for analyzing issues concerning antitrust regulation and competition in capacity investment. Nonetheless, the conditions under which a pure-strategy Cournot equilibrium exists are still under-explored. The aim of this paper is to show that a Cournot equilibrium exists under weaker conditions than imposed in the literature so far. Therefore, the class of cost and/or demand functions under which a Cournot equilibrium exists comes out to be richer than identified by the previous literature. Moreover, given the large number of applications of the Cournot paradigm, it is crucial that the conditions for existence of a Cournot equilibrium be well understood. Our study shows that, under the assumptions here stated, existence is relatively easy to prove as compared to existing studies.

Early work on the topic traces back to McManus (1962, 1964) and Roberts and Sonnenschein (1976). They find conditions on the primitives of the model, namely firm's technology and market inverse demand, that are sufficient to guarantee the equilibrium existence. In particular, they assume that competitors are all identical and have convex technologies.

Unlike these authors, Novshek (1985) shows that an equilibrium exists without requiring that firms be identical and under weaker assumptions on the technology of each firm. In turn, some stronger condition is imposed on the inverse demand function: each firm's marginal revenue is to decline with the aggregate output of the competitors.[1] As the author himself emphasizes, it is rather complicated to prove equilibrium existence in this setting.

More recently, Amir (1996, 2005) and Amir and Lambson (2000) have shown that the proof becomes nearly trivial as soon as one relies on the supermodular optimization methods, introduced by Topkis (1968, 1978, 1979) and further developed by Vives (1990), Milgrom and Roberts (1990) and Milgrom and Shannon (1991). However, in the presence of at least three firms $(n \geq 3)$, the Cournot oligopoly is not a supermodular game, in general, meaning that the results of this trend of the literature do not apply to more general settings where $n \geq 3$ (see Amir (1996, page 139, for a detailed explanation).

In this paper, we consider a n-firm Cournot market in which technologies are not necessarily identical across firms and the cost function of each firm needs to satisfy conditions that are somewhat weaker than imposed in McManus (1962, 1964). Moreover, we assume that the marginal revenue function weakly declines with the size of the output. This condition is weaker than the one imposed by Novshek, who takes it "to decline sufficiently much." We provide a relatively easy proof to show that, under these assumptions, a Cournot-equilibrium in pure strategies exists.

The paper is organized as follows. In section 2, we describe the model and some known results. In section 3, we present an example, section 4 is devoted to the main result of the paper and section 5 concluded.

[1]This method traces back to Selten (1970).

2 Model and two known results

A Cournot market game is described as follows: There is an industry composed of n firms producing an homogeneous good. Firm f produces the good in quantity $y_f \geq 0$ at a cost of $C_f(y_f)$, where $f \in \{1, ...n\}$ and $n \in \mathbb{N}_+ - \{1, 2\}$. We assume that the cost function $C_f(y)$, with $y \geq 0$, is not necessarily the same across competitors j, which means that technologies can be heterogeneous. The aggregate output is $y_{-f} + y_f$, where y_{-f} the output of all firms except that of firm f. Denoting further $P(q)$ the inverse demand function of the market, with q the aggregate demand and $q = y_{-f} + y_f$ at the equilibrium of the market, firm's f profit is written

$$\Pi_f(y_f, y_{-f}) = P(y_{-f} + y_f)y_f - C_f(y_f) \tag{1}$$

We say that $(y_1^*, y_2^*, \ldots, y_n^*)$ is a Cournot equilibrium if for each firm f, the following condition holds

$$P(y_{-f}^* + y_f^*)y_f^* - C_f(y_f^*) \geq P(y_{-f}^* + y)y - C_f(y), \forall y \geq 0, \tag{2}$$

where $y_{-f}^* = \sum_j y_j^* - y_f^*$.

Using the above setting, let us recall some results in the literature.

Theorem 1 *(McManus,1964). A n-firm Cournot equilibrium exists if:*

1. $\forall f \in \{1, 2, ...n\}$, $C_f(y) = C(y)$; *$C(y)$ is continuous, monotonically increasing and has nondecreasing incremental cost:*

$$C(y + x) - C(y) \geq C(y' + x) - C(y'), \forall y > y' \geq 0 \text{ and } x > 0$$

2. *$P(q)$ is non-increasing and such that $qP(q)$ is bounded. Moreover, $P(q)$ is upper-semi-continuous: $\forall q > 0$ and $\epsilon > 0$ there exists a number $\delta > 0$ such that if $|x - q| < \delta$ for some $x > 0$, then $P(x) < P(q) + \epsilon$.*

Theorem 2 *(Novshek, 1985). A n-firm Cournot equilibrium exists if:*

1. *$\forall f \in \{1, 2, \ldots, n\}$, $C_f(y)$ is nondecreasing and lower-semi-continuous, the latter property meaning that: $\forall y > 0$ and $\epsilon > 0$ there exists a number $\delta > 0$, for each $x > 0$ such that $|x - y| < \delta$ implies that $C(x) > C(y) - \epsilon$.*

2. *$P(.)$ satisfies the properties:*

   (a) *$P : \mathbb{R}_+ \to \mathbb{R}_+$ is continuous*

   (b) *There exists $0 < Z < \infty$ such that $P(Z) = 0$ and P is twice continuously differentiable and strictly decreasing on $[0, Z[$*

   (c) *For all $q \in [0, Z[$, $P'(q) + qP''(q) \leq 0$.*

Therefore Novshek shows that by changing the assumptions made by McManus, one can find that an equilibrium exists for non-identical firms (i.e. distinct technologies). The way he did was to weaken the assumptions on the cost function and make some stronger hypothesis on the demand. The reasoning of the proof of Novshek is similar to that of McManus, in that he found conditions under which the reaction correspondence is a monotonic function and so the equilibrium exists. However,the proof of Novshek's theorem is relatively long and complicated (as indicated by the author himself at page 90).

3 Example

We take in what follows a simple example, in which neither of the two theorems applies, to establish equilibrium existence.

Assume that the technology is such that

$$C_f(y) = \begin{cases} a_f \frac{y^2}{2}, \text{ if } y \leq c_f \\ y, \text{ if } y > c_f, \end{cases} .$$

where a_f and c_f are positive constants, allowed to be firm dependent and such that $a_f.c_f > 1$. Take further a standard demand function: a demand that is elastic and has constant price elasticity. It is defined by the following inverse demand curve:

$$P(q) = bq^{-\frac{1}{\varepsilon}},$$

with $\varepsilon = -\frac{p}{P'(q)q}$ the price-elasticity of demand, such that $\varepsilon > 1$, and $p = P(q)$ the market price that corresponds to the quantity demanded q.

Take some y such that $y > c_f$. Let x and y' such that $x = C(y+x) - C(y)$, $\frac{1}{a_f} < y' < c_f$ (y' exists by the assumption that $a_f.c_f > 1$) and $y' + x \leq c_f$. Assuming that the non-incremental cost condition of Theorem 1 is satisfied, one has

$$x = C(y+x) - C(y) \geq C(y'+x) - C(y') = a_f \frac{(y'^2 + x)}{2} - a_f \frac{y'^2}{2}.$$

Dividing by x, we get $1 \geq \frac{a_f \frac{(y'^2+x)}{2} - a_f \frac{y'^2}{2}}{x}$. When $x \to 0$, we obtain $1 \geq a_f.y'$, which contradicts the assumption. Hence, the cost function does not satisfy the non-incremental cost condition of Theorem 1 even if we assume, as the theorem does, that firms are identical: $a_f = a$ and $c_f = c$.

Moreover, since

$$P'(q) + qP''(q) = \frac{1}{\varepsilon^2} q^{-\frac{1}{\varepsilon}-1} > 0,$$

the demand function does not satisfy the condition (2c) of Theorem 2.

Even though neither the conditions of Theorem 1 nor those of Theorem 2, one can notice that the Cournot game is quasi-concave. First, the cost function $C_f(y)$ is non-concave. Second, by defining $R(y_f, y^*_{-f})$ the revenues function of firm f when it produces y_f and the other firms produce y^*_{-f}, one finds that $\frac{d^2 R(y_f, y^*_{-f})}{dy_f^2} < 0$, where

$$\frac{\partial^2 R(y_f, y^*_{-f})}{\partial y_f^2} = \frac{1}{\varepsilon} \left(y_f + y^*_{-f}\right)^{-\frac{1}{\varepsilon}-2} [(\frac{1}{\varepsilon} - 1)y_f - 2y^*_{-f}].$$

We show in this study that there are quasi-concave games in which a Cournot equilibrium exists, even though the technology does not satisfy the conditions of Theorem 1, or the demand function does not satisfy the conditions of Theorem 2, or both, as in the example just shown.

4 Result

In this section we state and prove our main result.

Theorem 3 *A n-firm Cournot equilibrium exists if:*

1. *$C_f(y)$ is such that, $\forall f \in \{1, 2, \ldots, n\}$,*
   (a) *It is nondecreasing and lower semi-continuous;*
   (b) *It satisfies the following weak non decreasing incremental cost condition: for each subset $\{y_1, y_2, y_3\}$, with $y_3 > y_2 > y_1$, there exists some $m > 1$ such that for each $x > 0$, and some $i < j$, with $i, j \in \{1, 2, 3\}$, the following condition holds:*
   $$C_f(y_j + x) - C_f(y_j) \geq C_f(y_i + x) - C_f(y_i) - x^m. \tag{3}$$
   (c) *$[C_f'(y_f)]^- \geq [C_f'(y_f)]^+$ for each y_f, where $[.]^-$ and $[.]^+$ stand for the left respectively the right derivative.*[2]
2. *$P(q)$ is such that*
   (a) *$P : \mathbb{R}_+ \to \mathbb{R}_+$ is continuous;*
   (b) *There exists $0 < Z < \infty$ such that $P(Z) = 0$ and P is twice continuously differentiable everywhere and strictly decreasing on $[0, Z[$;*
   (c) *The marginal revenue is a decreasing function in y_f.*

We will use for our proof another theorem, that we state below. Before doing that, we recall the definition of an f-graph continuous function, introduced by Dasgupta and Maskin (1986). The function $\Pi_f(y_f, y_{-j}) : A_f \times A_{-f} \to \mathbb{R}$ is said to be f-graph continuous if for all $y_f^0 \in A$ there exists another function $F_{-f} : A_{-f} \to A_f$, with $F_{-f}\left(y_{-f}^0\right) = y_f^0$ such that $\Pi_f(y_{-f}, F_{-f}(y_{-f}))$ is continuous at $y_{-f} = y_{-f}^0$. In the theorem below, we use as well the following definition:

$$\Pi_f^*(y_{-f}) = \underset{y_f}{Sup}\, \Pi_f(y_f, y_{-f}).$$

Theorem 4 *(Dasgupta and Maskin, 1986). If for all $f \in \{1, 2, ...n\}$ the payoff function $\Pi_f(y_f, y_{-f})$ satisfy:*

(i) $\Pi_f(y_f, y_{-f})$ is upper semi-continuous in y_{-f} and quasi-concave in y_f

(ii) $\Pi_f(y_f, y_{-f})$ is either f-graph continuous or such that the function $\Pi_f^*(y_{-f})$ is lower semi-continuous.

Then the game Γ_C has a Nash equilibrium in pure strategies.

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[2]Typically we are looking for cost functions defined on different intervals, for example $C(y) = \begin{cases} C^1(y), y \in I^1 \\ C^2(y), y \in I^2 \\ C^3(y), y \in I^3 \\ \ldots\ldots\ldots \\ C^s(y), y \in I^s \end{cases}$ such that $I^1 = [0, a^1], I^2 =]a^1, a^2], I^3 =]a^2, a^3], \ldots, I^s =]a^{s-1}, a^s]$ and the functions $C^j(.)$ are continuous and derivable in the interior of I^j. In the weak non decreasing incremental cost condition, the decreasing incremental cost condition may fails for y_1, y_2, but holds for y_1, y_3.

4.1 Proof of Theorem 3

Let us denote the Cournot game by $\Gamma_C = \{N, [0, Z], (\Pi_f)_{f\in N}\}$, where $N = \{1, 2, \ldots, n\}$ is the set of firms and $[0, Z]$ the set of strategies. A strategy of firm f is the level y_f of production. Denote $R_f(y_f, y_{-f}) = P(y_{-f} + y_f)y_f$ the market revenues of f when it produces y_f and the other firms produce y_{-f}. We prove the theorem in three steps.

The first step is to consider the virtual game Γ_0, defined as $\Gamma_0 = \{N, [0, Z], (R_f)_{f\in N}\}$. Using conditions (2a) - (2c) the revenue function $R_f(y_f, y_{-f})$ is twice continuously differentiable on $[0, Z[$. Thus, for any y_f on this interval, one has

$$\frac{\partial R_f}{\partial y_f}(y_f, y_{-f}) = P(y_{-f} + y_f) + y_f P'(y_{-f} + y_f) \tag{4}$$

and

$$\frac{\partial R_f^2}{\partial y_f^2}(y_f, y_{-f}) = 2P'(y_{-f} + y_f) + y_f P''(y_{-f} + y_f). \tag{5}$$

Moreover, using (4)

$$\frac{\partial R_f}{\partial y_f}(0, y_{-f}) = P(y_{-f}) \geq 0 \text{ and } \frac{\partial R_f}{\partial y_f}(Z, y_{-f}) = ZP'(y_{-f} + Z) \leq 0. \tag{6}$$

From (4), (5), (6) and condition (2.c) of the Theorem, the virtual game Γ_0 is concave and so the best response function $\widehat{y}_f(y_{-f})$ is unique for each $f \in N$. Moreover, $\widehat{y}_f(y_{-f}) \in]0, Z[$.

The second step is to rely on Γ_0 to show that the game Γ_{C_f} have the property that the best reply correspondence of each firm is unique.

Suppose on the opposite that there are two best replies in game Γ_C. This implies the existence of two maxima of the function $\Pi_f(., y_{-f})$, that we denote ${y_f}^1$ and ${y_f}^3$, and one minimum y_f^2, such that ${y_f}^1 < y_f^2 < {y_f}^3$. As the curve of $\Pi_f(., y_{-f})$ is under the curve of $R_f(., y_{-f})$, each of the three extreme points of $\Pi_f(., y_{-f})$ is reached at a level of y_f that is strictly lower than $\widehat{y}_f\,(y_{-f})$. Using as well the information that $\widehat{y}_f(y_{-f}) \in]0, Z[$, it means that the three local extremum points belong to the interval $]0, Z[$. Using the property that $C(., y_{-f})$ is lower semi-continuous, the three local extreme points are such that

$$\begin{aligned}
\frac{\partial R_f}{\partial y_f}(y_f^1, y_{-f}) &= \left[C'(y_f^1)\right]^+ \\
\frac{\partial R_f}{\partial y_f}(y_f^2, y_{-f}) &= \left[C'(y_f^2)\right]^- \\
\frac{\partial R_f}{\partial y_f}(y_f^3, y_{-f}) &= \left[C'(y_f^3)\right]^+.
\end{aligned}$$

Moreover, by condition (2.c) of the theorem, $\frac{\partial R_f}{\partial y_f}(y_f, y_{-f})$ is decreasing with y_f. Using this property and the definition of ${y_f}^1$, ${y_f}^2$ and ${y_f}^3$, it implies that

$$\left[C'(y_f^3)\right]^+ < \left[C'(y_f^2)\right]^- < \left[C'(y_f^1)\right]^+. \tag{7}$$

Apply now the weak non decreasing incremental cost condition (1b) of the theorem to the subset $\{{y_f}^1, {y_f}^2, {y_f}^3\}$. There exists some $m > 1$, substitute x by $x_k = \frac{1}{k}$ in (3), where $k \in \mathbb{N}$ and assume that the inequality is satisfied infinitely with ${y_f}^1, {y_f}^2$. Then

$$C_f({y_f}^2 + x_k) - C_f({y_f}^2) \geq C_f({y_f}^1 + x_k) - C_f({y_f}^1) - (x_k)^m.$$

Dividing by x_k,

$$\frac{C_f({y_f}^2+x_k)-C_f({y_f}^2)}{x_k} \geq \frac{C_f({y_f}^1+x_k)-C_f({y_f}^1)}{x_k} - (x_k)^{m-1}$$

As $m-1>0$ (by assumption), one obtains for $k \longrightarrow \infty$ that

$$\left[C'(y_f^2)\right]^+ \geq \left[C'(y_f^1)\right]^+.$$

However, using it together with

$$\left[C'(y_f^2)\right]^- \geq \left[C'(y_f^2)\right]^+$$

from condition (1c), we get a contradiction of (7). Therefore, the hypothesis that there exist two best replies in the game Γ_{C_f} for some f leads to a contradiction. Hence the game is quasi-concave.

The third step is to use Theorem 4 to show that the equilibrium exists. Since $R(.,.)$ is continuous, and every best reply belongs to the interval $]0,Z[$, we need to show that the function defined as

$$y_{-f} \rightarrow \Pi_f^*(y_{-f}) = \underset{y_f\in[0,Z]}{Sup}\ \Pi_f(y_f,y_{-f}),$$

is lower semicontinuous. To do that we have just to show that its epigraph $Ep(\Pi_f^*)=\{(y_{-f},\xi) : \Pi_f^*(y_{-f})\leq\xi\}$ is closed. Let $(y_{-f}^k,\xi^k)_k$ be a sequence where each of its elements belongs to the epigraph $Ep(\Pi_f^*)$. Assume that $(y_{-f}^k,\xi^k)_k$ converges to (y_{-f},ξ). From the upper semi continuity of the function $\Pi_f(.,y_{-f})$ and the compactness of the subset $[0,Z]$, for each y_{-f}, the supremum $\underset{y_f\in[0,Z]}{Sup}\ \Pi_f(y_f,y_{-f})$ is reached at some level $y_f^{max}\in[0,Z]$. Moreover we have shown that y_f^{max} is an interior element of the strategy set $[0,Z]$, that is $y_f^{max}\in]0,Z[$.

For the sequence $(y_{-f}^k,\xi^k)_k$, we have

$$\underset{y_f\in[0,Z]}{Sup}\ \Pi_f(y_f,y_{-f}^k)=\Pi_f(y_f^{max,k},y_{-f}^k)=P(y_{-f}^k+y_f^{max,k})y_f^{max,k}-C_f(y_f^{max,k})\leq\xi^k,$$

where $y_{-f}^k=\sum_{j=1,\ldots,n;j\neq f} y_f^k$.

When $k\rightarrow\infty$, we get

$$\begin{aligned}\lim_{k\rightarrow\infty}\ \underset{y_f\in[0,Z]}{Sup}\ \Pi_f(y_f,y_{-f}^k) &= \lim_{k\rightarrow\infty}\ \left(\Pi_f(y_f^{max,k},y_{-f}^k)\right)=\lim_{k\rightarrow\infty}\ \left(P(y_{-f}^k+y_f^{max,k})y_f^{max,k}-C_f(y_f^{max,k})\right)\\ &= P(y_{-f}^k+y_f^{max})y_f^{max}-\lim_{k\rightarrow\infty}\ \left(C_f(y_f^{max,k})\right)\leq\xi. \qquad (8)\end{aligned}$$

Now assume that

$$P(y_{-f}+y_f^{max})y_f^{max}-C_f(y_f^{max})>\xi. \qquad (9)$$

Then, using (8), it follows that

$$\lim_{k\rightarrow\infty}\ \left(C_f(y_f^{max,k})\right)>C_f(y_f^{max}),$$

which is possible only if the best strategy is reached to the right of the discontinuity of the cost function $C_f(\cdot)$. Under our assumption on the cost function, this is impossible and so it contradicts hypothesis (9). Then, the conditions of Theorem 4 are satisfied and an equilibrium in pure strategies exists.

5 Discussion

The proof shows that the existence of Cournot equilibrium follows from an application of the standard existence theorem for concave games. Let us compare now our theorem with previous results in the literature. Unlike McManus (1964), the competitors are not required to have the same cost function and the condition on the incremental cost is weakened as well. First, given the subset $\{y_1, y_2, y_3\}$, the incremental cost condition does not need to hold for all subsets $\{i, j\}$ of $\{1, 2, 3\}$. Indeed, if the incremental cost condition is satisfied for the subset $\{1, 2\}$, then it is not necessary to be satisfied for $\{1, 3\}$ as well. Moreover, because $x^m > 0$, it means that even for the subset $\{i, j\}$ the condition is weaker than the standard incremental cost condition of McManus (1964). Another way to see this result is by using the remark that, under condition (3), if the cost function is convex on a small interval $[y_1, y_2]$, then the Cournot equilibrium exists without imposing any restriction on the shape of $C_f(.)$ on the larger interval $[y_1, y_3]$. This result shows that a Cournot equilibrium exists for some technologies that exhibit different shapes of the cost function. In our previous example, this is the case of technologies that are convex for low productions and linear for high productions.

Furthermore, the comparison between our result and that of Novshek (1985) shows that we have weakened the condition (2c) on the inverse demand function, ”at the expense” of some stronger condition on the cost function ((1.a) and (1.b) in Theorem 3 as compared to condition (1) in Theorem 2). This allowed us to prove, with a relatively easy proof, the existence of a Cournot equilibrium for a larger class of demand functions than in his study. Using our previous example, the equilibrium exists for a demand function whose elasticity is constant and superior to 1, even though the conditions of Novshek's study are not satisfied.

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