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Abstract. An agent is said to be partially honest if he or she weakly prefers an outcome at a strategy profile with his truthful strategy than an outcome at a strategy profile with his false strategy, then this player must prefer strictly the "true" strategy profile to the "false" strategy profile. In this paper we consider an exchange economy with single peaked preferences. With many agents ($n \geq 3$), if there exists at least one partially honest agent, we prove that any solution of the problem of fair division satisfying unanimity is Nash implementable.

Keywords: Nash implementation; Partial honesty; Single-peaked preferences.
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1 Introduction

Given some desired outcomes, whether there exists game form for which the strategic interactions require agents to choose actions that give the desired outcomes. This is the aim of the implemenation theory. In standard framework of Nash implementation, Maskin (1977, 1999) was the first who shows that any social choice correspondence (SCC) satisfying Maskin monotonicity and no veto power condition can be implemented in Nash equilibria. The Maskin monotonicity condition is a necessary condition for implementation, but the no veto power condition is not. For a full characterization, Maskin's Theorm was extended by Moore and Repullo (1990), Dutta and Sen (1991), Sjöström (1991) and Danilov (1992) [1] . Recently, Matsushima (2008) and Dutta and Sen (2010) introduce,

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[1] The Maskin's results are also extended by several authors not cited here.

in two different frameworks, the notion of honesty as an element of behavioral economic perspectives. Matsushima (2008) considers a social choice function (SCF) which assigns to each possible *vNM preference* profile a *lottery* over the basic set of outcomes. He assumes that *all* players have intrinsic preference for honesty oriented in lexicographical way on a complete information environment with small fines. He shows that when there exists at least three players in the mechanism, any SCF is implementable in the iterative elimination of strictly dominated strategies, and in Nash equilibria. Dutta and Sen (2010) consider Nash implementation problems with the assumption that there is at least one partially honest agent who not only has the standard self-interested preference on alternatives but also has an intrinsic preference for honesty. They assume that a player is partially honest if she prefers weakly an outcome of a strategy profile with his truthful strategy than an outcome of a strategy profile with his false strategy, then this player must prefer strictly the "true" strategy profile than the "false" strategy profile. The identity of this player is not known by the planner. In this setting, they show that when there exist at least three players, any SCC satisfying no veto power can be implemented in Nash equilibria. In a domain of strict orders, they also provide necessary and sufficient conditions for implementation in the two-player case when there is exactly one partially honest individual and when both individuals are partially honest. These latter results were extended by Lombardi (2010) to the domain of weak orders.

We consider an exchange economy with single peaked preferences. If there exists at least one partially honest agent, we prove that all solutions of the problem of fair division satisfying unanimity are Nash implementable.

2 Notations and definitions

Let A be the set of alternatives, and let $N = \{1, ..., n\}$ be the set of individuals, with generic element i. Each individual i is characterized by a preference relation R_i defined over A, which is a complete, transitive, and reflexive relation in some class $\Re_i$ of admissible preference relations. Let $\Re = \Re_1 \times ... \times \Re_n$. Let $\mathscr{D} \subset \Re$ be a domain. An element $R = (R_1, ..., R_n) \in \mathscr{D}$ is a preference profile. The relation R_i indicates the individual i's preference. For $a, b \in A$, the notation aR_ib means that the individual i prefers weakly a to b. The asymmetrical and symmetrical parts of R_i are denoted respectively by P_i and $\sim_i$.
A social choice correspondence (SCC) F is a function from $\Re$ into $2^A \setminus \emptyset$, that associates with every R a nonempty subset of A. For all $R_i \in \Re_i$ and all $a \in A$, the lower contour set for agent i at alternative a is noted by: $L(a, R_i) = \{b \in A \mid aR_ib\}$. The strict lower contour set is denoted by $LS(a, R_i) = \{b \in A \mid aP_ib\}$.

Now, we model an environment for partial honesty. As Dutta and Sen (2009), we assume that an honest player's preference for honesty is *lexicographic*. Let $S_i = \Re \times C_i$ be the set of strategy profiles for a player i , where C_i denotes the other components of the strategy space (which depends on individual preferences, social states,...). Let $S = S_1 \times ... \times S_n$ be a set of strategy profiles. The elements of S are denoted by $s = (s_1, ..., s_n)$. For each $i \in N$, and $R \in \mathscr{D}$, let $\tau_i(R) = \{R\} \times C_i$ be the set of truthful messages of agent i. We denote by $s_i \in \tau_i(R)$ a truthful strategy as player i is reporting the true preference profile. We extend a player's ordering over A to an ordering over strategy space S. This is because, the players' preference between being honest and dishonest depends on strategies that the others played and of the outcomes which they obtain. Let $\succeq_i^R$ be the preference of player i over S in preference profile R. The asymmetrical and symmetrical parts of R_i are denoted respectively by $\succ_i^R$ and $\sim_i^R$. Let Γ be a mechanism (game form) represented by the pair (S, g), where $g : S \to A$ is a payoff function.

Definition 1 *A player i is partially honest if for all preference profile $R \in \mathscr{D}$ and $(s_i, s_{-i}), (s'_i, s_{-i}) \in S$,*

(i) when $g(s_i, s_{-i})R_i g(s'_i, s_{-i})$ and $s_i \in \tau_i(R)$, $s'_i \notin \tau_i(R)$, then $(s_i, s_{-i}) \succ_i^R (s'_i, s_{-i})$.

(ii) In all other cases, $(s_i, s_{-i}) \succeq_i^R (s'_i, s_{-i})$ iff $g(s_i, s_{-i})R_i g(s'_i, s_{-i})$.

A Nash equilibrium of the game $(\Gamma, \succeq^R)$ is a vector of strategies $s \in S$ such that for any i, $g(s)R_i g(b_i, s_{-i})$ for all $b_i \in S_i$, i.e. when the other players choose s_{-i}, the player i cannot be better off by deviating from s_i. Given $N(g, \succeq^R, S)$ the set of Nash equilibria of the game $(\Gamma, \succeq^R)$, a mechanism $\Gamma = (S, g)$ implements a SCC F in Nash equilibria if for all $R \in \mathscr{D}$, $F(R) = g(N(g, \succeq^R, S))$. We say that a SCC F is implementable in Nash equilibria if there is a mechanism which implements it in these equilibria.

Assumption (A): There exists at least one partially honest individual and this fact is known to the planner. However, the identity of this individual is not known to her.

Definition 2 *(Unanimity). An SCC F satisfies unanimity if for any $a \in A$, any $R \in \Re$, and for any $i \in N$, $L(a, R_i) = A$ implies $a \in F(R)$.*

3 Exchange economy with single-peaked preferences

There is an amount $\Omega \in \mathbb{R}_{++}$ of certain infinitely divisible good that is to be allocated among a set $N = \{1, ..., n\}$ of n agents. The preference of each agent $i \in N$ is represented by a continuous and single-peaked preference relation R_i over $[0, \Omega]$ (the asymmetrical part is written P_i and the symmetrical part $\sim_i$). For all $x_i, y_i \in [0, \Omega]$, $x_i R_i y_i$ mean that, for the agent i, to consume a share x_i is as good as to consume the quantity y_i. A feasible allocation for the economy (R, Ω) is a vector $x \equiv (x_i)_{i \in N} \in \mathbb{R}^n_+$ such that $\sum_{i \in N} x_i = \Omega$ and X is the set of the feasible allocations. We note that the feasible allocations set is $X = [0, \Omega] \times ... \times [0, \Omega]$. Thus, $L(x, R_i) = X$ is equivalent to $L(x_i, R_i) = [0, \Omega]$. For the set $L(x, R_i) = X$, $x R_i y$ for all $y \in X$ implies that $x_i R_i y_i$. We note that the free disposability of the good is not assumed. A preference relation R_i is *single peaked* if there is a number $p(R_i) \in [0, \Omega]$ such that for all $x_i \in [0, \Omega]$ if $y_i < x_i \leq p(R_i)$ or $p(R_i) \leq x_i < y_i$, then $x_i P_i y_i$. We call $p(R_i)$ the peak of R_i.

The class of all single-peaked preference relations is represented by $\mathscr{D}_{sp} \subseteq \mathscr{D}$. For $R \in \mathscr{D}_{sp}$, let $p(R) = (p(R_1), ..., p(R_n))$ be the profile of peaks (or of preferred consumptions). A single peaked preference relation $R_i \in \mathscr{D}_{sp_i}$ is described by the function $r_i : [0, \Omega] \to [0, \Omega]$ which is defined as follows: $r_i(x_i)$ is the consumption of the agent i on the other side of the peak which is indifferent to x_i (if it exists), else, it is 0 or Ω. Formally, if $x_i \leq p(R_i)$, then, $r_i(x_i) \geq p(R_i)$ and $x_i \sim_i r_i(x_i)$ if such a number exists or $r_i(x_i) = \Omega$ otherwise; if $x_i \geq p(R_i)$, then, $r_i(x_i) \leq p(R_i)$ and $x_i \sim_i r_i(x_i)$ if such a number exists or $r_i(x_i) = 0$ otherwise. Let us introduce some known correspondences.

Pareto solution (Pro): $P(R) = \{x \in X : \nexists x' \in X$ such that for all $i \in N$, $x'_i R_i x_i$, and for some $i \in N$, $x'_i P_i x_i\}$.

The Pareto correspondence is the solution which associates each economy with its feasible allocation set such that there does not exist any other feasible allocation that all agents prefer weakly and at least one prefers strictly.

The No-Envy correspondence (NE): $NE(R) = \{x \in X$ if $x_i R_i x_j$ for all $i, j \in N\}$.

The no-envy correspondence selects the allocations at which no agent prefers some others consumption to his own. Formally:

Individually Rational Correspondence from Equal Division (I_{ed}):$I_{ed}(R) = \{x \in X : x_i R_i (\Omega/n)$ for all $i \in N\}$.

Proportional correspondence (Pro): Let $R \in \mathscr{D}_{sp}$, $x \in Pro(R)$ if $x \in X$ and (i) when $\sum_{i \in N} p(R_i) \geq 0$, and $\exists \lambda \in \mathbb{R}_+$ s.t. $\forall i \in N$, $x_i = \lambda p(R_i)$; and (ii) when $\sum_{i \in N} p(R_i) = 0$, $x = (\Omega/n, ..., \Omega/n)$.

This solution selects the allocation for which each agent obtain a share proportionally to its preferred consumption if at least one preferred consumption is positive, and an average share if all preferred consumptions equal zero.

Symmetrically Proportional correspondence ($SPro$): Let $R \in \mathscr{D}_{sp}$, $x \in SPro(R)$ if $x \in X$ and (i) when $\sum_{i \in N} p(R_i) \geq \Omega$, and $\exists \lambda \in \mathbb{R}_+$ s.t. $\forall i \in N$, $x_i = \lambda p(R_i)$; and (ii) when $\sum_{i \in N} p(R_i) \leq \Omega$, $\exists \lambda \in \mathbb{R}_+$ s.t. $\Omega - x_i = \lambda(\Omega - p(R_i))$ for all $i \in N$.

This solution is quite simply a symmetricized version of the proportional solution where the units of the good are treated symmetrically above the preferred consumptions if the sum of latter is less than the amount Ω, or below it (the preferred consumptions) if its sum is greater than Ω.

Equal-Distance correspondence (Dis): Let $R \in \mathscr{D}_{sp}$, $x \in Dis(R)$ if $x \in X$ and (i) when $\sum_{i \in N} p(R_i) \geq \Omega$, $\exists d \geq 0$ s.t. $\forall i \in N$, $x_i = max\{0, p(R_i) - d\}$; and (ii) when $\sum_{i \in N} p(R_i) \leq \Omega$, $\exists d \geq 0$ s.t. $x_i = p(R_i) + d$ for all $i \in N$.

This solution is based on comparing distances from preferred consumptions. It selects the allocation at which all agents are equally distant from their preferred consumptions if the sum of these latter is less than the amount Ω, otherwise, each agent obtain the maximum of zero and a share given by his preferred consumptions minus a distance.

Equal-Sacrifice correspondence (Sac): Let $R \in \mathscr{D}_{sp}$, $x \in Sac(R)$ if $x \in X$ and (i) when $\sum_{i \in N} p(R_i) \geq \Omega$, $\exists \sigma \geq 0$ s.t. $\forall i \in N$, $r_i(x_i) - x_i \leq \sigma$, strict inequality holds only if $x_i = 0$; and (ii) when $\sum_{i \in N} p(R_i) \leq \Omega$, $\exists \sigma \geq 0$ s.t. $x_i - r_i(x_i) = \sigma$ for all $i \in N$.

This solution is based on the idea of the measurement of "the sacrifice" at allocation x by the size of agent i's upper contour set at x_i. It selects efficient allocations at which sacrifices are equal across agents when the sum of preferred consumptions is less than the amount Ω, otherwise the agents which would have negative consumptions, get zero.

In a standard domain of exchange economy with single-peaked preferences, Thomson (1990, 2010) examined the implementation problem of many solutions of the problem of fair division. For monotonic correspondences, he utilised different techniques developed in the literature. By checking Maskin's conditions, he found that only Pareto correspondence (P) satisfies no veto power condition and hence it is implementable by Maskin's Theorem (19977, 1999). The family (monotonic solutions) of No-Envy correspondence (NE), Individually rational correspondence from equal division (I_{ed}), ($NE \cap I_{ed}$) correspondence, ($NE \cap P$) correspondence and ($NE \cap I_{ed}$) correspondence fail to satisfy no veto power condition. Therefore Maskin's theorem can not give an information on their implementability. For this, Thomson exploited a strong version of Maskin monotonicity, called *strong monotonicity*, proposed by Daniliv (1992) and generalized by Yamato (1992). He showed that this condition is satisfied by (P), (NE), (I_{ed}) and ($NE \cap I_{ed}$) correspondences. Thus, he succeed to implement latter solutions by Yamato's theorem (1992). However, he proved that strong monotonicity does not hold for ($NE \cap P$) correspondence and ($NE \cap I_{ed}$) correspondence. He concluded that strong monotonicity is not stable under intersection. To implement these latter correspondences, Thomson imposed difficult conditions.

Doghmi and Ziad (2008b) used new sufficient conditions of strict monotonicity, (strict weak) no veto power and unanimity developed in Doghmi and Ziad (2008a). They showed that these conditions are satisfied by the different solutions cited above. Moreover, they illustrated that these conditions are very simple and stable under intersection. This tool of stability is very useful to detect directly the implementabilty of the correspondences produced by intersection. The simplicity of these conditions appears in the fact that, in this domain under consideration, Unanimity is verified by all solutions, strict weak no veto power is satisfied independently to studied solution, and strict monotonicity becomes equivalent to Maskin monotonicity. Doghmi and Ziad (2008b) gave a full characterization in this domain by showing that just Maskin monotonicity alone is necessary and sufficient for implementation. Thus, by this very easy condition compared to the different techniques used by Thomson (1990, 2010), they solved definitively the problem of implementation in a domain of exchange economies with single peaked preferences. With the honesty assumption, the following is the main result of the paper.

Theorem 1 . *Let $n \geq 3$ and suppose Assumption (A) holds. Any SCC satisfying unanimity is implementable in Nash equilibria. In particulary (Pro), (NE), (I_{ed}), ($SPro$), (Dis), (Sac) or any intersection of them are implementable in Nash equilibria.*

Proof. By Unanimity and Lemma 1 of Doghmi and Ziad (2008b), the strict weak no veto power condition is satisfied independently of any studied solution.

By proposition 1 in the appendix, we compete the proof. Q.E.D.

4 Appendix

In this section, we present the sufficient conditions that characterize the family of social choice correspondences that can be implemented with partially honest agents.

We introduce the following weak version of no veto power condition.

Definition 3 *(Strict weak no veto power). An SCC F satisfies strict weak no veto power if for i, $R \in \Re$, and $a \in F(R)$, for $R' \in \Re$, $b \in LS(a, R_i) \subseteq L(b, R'_i)$ and $L(b, R'_j) = A$ for all $j \in N \backslash \{i\}$, then $b \in F(R')$.*

We have the following proposition.

Proposition 1 . *Let $n \geq 3$ and suppose Assumption A holds. If an SCC F satisfies strict weak no veto power and unanimity, then F can be implemented in Nash equilibria.*

Proof. Let $\Gamma = (S, g)$ be a mechanism which is defined as follows: For each $i \in N$, let $S_i = \mathscr{D} \times A \times \mathbb{N}$, where $\mathbb{N}$ consists of the nonnegative integers. The generic element of strategic space S_i is noted by: $s_i = (R^i, a^i, m^i)$. Each agent announces a preference profile, an optimal alternative and nonnegative integer. The function g is defined as follows:

Rule 1: If for each $i \in N$, $s_i = (R, a, 1)$ and $a \in F(R)$, then $g(s) = a$.
Rule 2: If for some i, $s_j = (R, a, m)$ for all $j \neq i$, $a \in F(R)$ and $s_i = (R^i, a^i, m^i) \neq (R, a, m)$, then:

$$g(s) = \begin{cases} a^i & \text{if} \quad a^i \in LS(a, R_i) \neq \emptyset, \\ a & \text{otherwise.} \end{cases}$$

Rule 3: In any other situation, $g(s) = a^{i^*}$, where i^* is the index of the player of which the number m^{i^*} is largest. If there are several individuals who satisfy this condition, the smallest index i will be chosen.

Let us show that $F(R) = g(N(g, \succeq^R, S))$. The proof contains two steps:

Step 1. For all $R \in \mathscr{D}$, $F(R) \subseteq g(N(g, \succeq^R, S))$.

Let $R \in \mathscr{D}$ and $a \in F(R)$. For each $i \in N$, let $s_i = (R, a, 1)$. Then, by definition of g, we have $s \in N(g, \succeq^R, S))$ and $g(s) = a$.

Step 2. For all $R \in \mathscr{D}$, $g(N(g, \succeq^R, S)) \subseteq F(R)$.

Let $s \in N(g, \succeq^R, S)$, we show that $g(s) \in F(R)$. For that, we study the various possibilities of writing the profile of strategies $s = (s_1, s_2, ..., s_n)$.

Case a: Suppose there exists $(R', a, m) \in \Re \times A \times \mathbb{N}$, with $a \in F(R')$, such that s is defined by $s_i = (R', a, m)$ for any $i \in N$. Then, by rule 1, $g(s) = a$. Let $R' = R$, $a \in F(R)$. Assume that $R' \neq R$. Let i be a partially honest individual. The individual i can deviate to the truthful announcement of R by playing $\widetilde{s}_i = (R, a, m') \in \tau_i(R)$ with $m' > m$. Then, by rule 2, $a = g(\widetilde{s}_i, s_{-i}) R_i g(s_i, s_{-i}) = a$. By partially-honest assumption, $(\widetilde{s}_i, s_{-i}) \succ_i^R (s_i, s_{-i})$, therefore $s \notin N(g, \succeq^R, S)$, a contradiction. We conclude that $s_i \in \tau_i(R)$ for all $i \in N$. Therefore, $R' = R$ and hence $g(s) = a \in F(R)$.

Case b: $s = (s_1, s_2, ..., s_n)$. Assume there is $i \in N$, $R' \in \mathscr{D}$ and $a \in A$ such that $a \in F(R')$. For all $j \neq i$, $s_j = (R', a, m)$ and $s_i = (R'^i, a^i, m^i) \neq s_j$. Then, by rule 2,

$$g(s) = \begin{cases} a^i & \text{if} \quad a^i \in LS(a, R'_i) \neq \emptyset, \\ a & \text{otherwise.} \end{cases}$$

Subcase b1: $g(s) = a^i \neq a$

By definition of g, we have $a^i \in LS(a, R'_i) \neq \emptyset$. Take any $b \in LS(a, R'_i) \neq \emptyset$ and a deviation $\widetilde{s}_i$ for player i such that $\widetilde{s}_i = (\widetilde{R}, b, \widetilde{m})$ with $\widetilde{m} > m$. By Rule 2, $g(\widetilde{s}_i, s_{-i}) = b$. Since $s \in N(g, \succeq^R, S)$, then $a^i = g(s) R_i g(\widetilde{s}_i, s_{-i}) = b$, i.e., $b \in L(a^i, R_i)$. Therefore $a^i \in LS(a, R'_i) \subseteq L(a^i, R_i)$. (1)

Next, for any other deviation $j \neq i$ and any $c \in A$, let $\widetilde{s}_j = (\widetilde{R}, c, \widetilde{m})$ be a deviation, where $\widetilde{m}$ is the unique greatest integer in the profile $(\widetilde{s}_j, s_{-j})$. By rule 3, $g(\widetilde{s}_j, s_{-j}) = c$. Since $s \in N(g, \succeq^R, S)$, we have $a^i = g(s) R_i g(\widetilde{s}_j, s_{-j}) = c$. Therefore, for all $j \neq i$, $A \subseteq L(a^i, R_j)$. (2)

From (1), (2) and by strict weak no veto power, we have $a^i \in F(R)$.

Subcase b2: $g(s) = a$

By the same reasoning as in Case a, $a \in F(R)$.

Case c: $s = (s_1, s_2, ..., s_n)$: $\exists k_1, k_2, k_3$ where $s_{k_1} \neq s_{k_2}$, $s_{k_1} \neq s_{k_3}$, $s_{k_2} \neq s_{k_3}$, $g(s) = a_l$: m_l is the maximum of the integers m. By Rule 3 and Unanimity, $g(s) \in F(R)$. Q.E.D.

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