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Nash equilibria in nonsymmetric singleton congestion games with exact partition

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Abstract We define a new class of games, which we qualify as congestion games with exact partition. These games constitute a subfamily of singleton congestion games for which the players are restricted to choose only one strategy, but they each possess their own utility function. The aim of this paper is to develop a method leading to an easier identification of all Nash equilibria in this kind of congestion games. We also give a new proof establishing the existence of a Nash equilibrium in this type of games without invoking the potential function or the finite best-reply property.

Keywords Singleton congestion games · Nash equilibria · Potential function · Finite best-reply property.

Mathematics Subject Classification (2000) C72

1 Introduction

A central concept in game theory is the notion of an equilibrium. One of the most widely used solution concepts for noncooperative games is the one of Nash equilibrium. A Nash equilibrium is a state in which no player can

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improve his objective by unilaterally changing his strategy. Rosenthal was the first to introduce, in 1973, a special class of noncooperative games, widely known as congestion games. These games lie in the intersection between, on the one hand, game theory, as they constitute noncooperative games, and on the other hand, computer science because they can model diverse phenomena such as processor scheduling, routing, and network design.

In congestion games, players' strategies consist of subsets of resources, and the utility of a player depends only on the number of players choosing the same or some overlapping strategy. The utility a player derives from a combination of resources is the sum of the payoffs associated with each resource included in his choice. Rosenthal (1973) shows that congestion games always possess Nash equilibria. This line of work has been continued by Monderer and Shapley (1996) who established a connection between potential games and congestion situations. An excellent survey of the related literature can be found in Voorneveled et al (1999). However, Konishi et al (1997) and Quint and Shubik (1994) consider that congestion games do not admit (in general) a potential function, but are likely to admit a Nash equilibrium in pure strategies.

In 1996, Milchtaich introduced a new class of congestion games, namely the congestion games with player-specific payoff functions, also called nonsymmetric singleton congestion games or singleton congestion games, for short. Each player has individual nonincreasing payoff functions and is allowed to choose any resource but must choose exactly one. In other means, the players' strategies are singletons and the payoff functions not only are decreasing but at the same time specific to each player. Milchtaich demonstrated that such games do not generally admit an exact potential function, nonetheless, each game in this class admits at least one Nash equilibrium that can be rehashed as a terminal point of a particular improvement dynamic. Particularly, he showed that such games possess the finite best-replay property (FBRP)[1] and an obvious consequence of the existence of the FBRP is the existence of a Nash equilibrium. Even so, the mechanism of the FBRP allows to construct only one Nash equilibrium. If we want to find all Nash equilibria, we have to repeat the FBRP process and this, maybe to the infinity ...! Additionally, we have to note that, until now, it does not exist in the literature a mathematical formula of the potential function that allows to establish at least one Nash equilibrium in such games. However, it seems both useful and interesting to extend and at the same time simplify the analysis initiated by Milchtaich, by taking as a starting point the following question: Is it possible to propose an alternative mechanism that allows to describe all Nash equilibria, at least in some particular cases of singleton congestion games?

In this paper, we give an answer to the above question. Specifically, we examine a special case of singleton congestion games, which we qualify as

[1] Paths in which in each step the unique deviator shifts to a strategy which is a best reply against the strategies played by the other players are called best-reply paths. A best-reply strategy need not be unique. If players deviate only when the strategy they are currently playing is not a best-reply strategy then the path is a best-reply improvement path (Milchtaich (1996)).

congestion games with exact partition. Our aim is to prove that such games possess at least one Nash equilibrium, by direct and constructive proofs, without using either the notion of the potential function or the FBRP, to show how to compute all equilibria and to give their structure using a simple and direct method. Note that the characterization of the set of all equilibria, beyond its theoretical interest, can be very useful when we have to choose between these equilibria on the basis of performance criteria such as social optimality, or to explore intrinsic proprieties of the game such as the price of anarchy[2] (Koutsoupias and C. H. Papadimitriou (1999)). The rest of this paper is organized as follows: Section 2 provides the main definitions and notations of congestion games, section 3 provides the result and section 4 concludes the paper.

2 Definitions and notations

A game (in strategic form) is defined by a tuple $\Gamma = (N, (S_i)_{i \in N}, (u_i)_{i \in N})$, where $N = \{1, 2, \ldots, n\}$ is a set of n players, S_i a finite set of strategies available to player i and $u_i : S = S_1 \times \ldots \times S_n \to \mathbb{R}$ is the utility function of player i. The set S is the strategy space of the game, and its elements are the (strategy) profiles. For a profile $\sigma = (\sigma_i)_{i \in N}$ on S, we will use the notation σ_{-i} to stand for the same profile with i's strategy excluded, so that (σ_{-i}, σ_i) forms a complete profile of strategies. A (pure) Nash equilibrium of the game Γ is a profile σ^* such that each σ_i^* is a best-reply strategy: For each player $i \in N$, $u_i(\sigma^*) \geq u_i(\sigma_i, \sigma_{-i}^*)$, for all $\sigma_i \in S_i$. Thus, no player can benefit from unilaterally deviating from his strategy. In a standard congestion game, defined by Rosenthal (1973), we are given a finite set $R = \{1, \ldots, m\}$ of m resources. A player's strategy is to choose a subset of resources among a family of allowed subsets: $S_i \subseteq 2^R$, for all $i \in N$. A payoff function $d_r : \{1, \ldots, m\} \to \mathbb{R}$ is associated with each resource $r \in R$, depending only on the number of players using this resource. For a profile σ and a resource r, the congestion on resource r (i.e. the number of players using r) is defined by $n_r(\sigma) = |\{i \in N : r \in \sigma_i\}|$. The vector $(n_1(\sigma), \ldots, n_m(\sigma))$ is the congestion vector corresponding to σ. The utility of player i from playing strategy σ_i in profile σ is given by $u_i(\sigma) = \sum_{r \in \sigma_i} d_r(n_r(\sigma))$. Rosenthal shows that every congestion game possesses at least one Nash equilibrium by considering the exact potential function $P : S \to \mathbb{N}$ with $P(\sigma) = \sum_{r \in R}^{m} \sum_{j=1}^{n_r(\sigma)} d_r(j), \forall \sigma \in S$ [3].

An extension of congestion games is the monotone nonsymmetric singleton congestion games (singleton congestion games for short), which can be seen as

[2] When utilities are replaced by costs, the price of anarchy of a game is the ratio of the social cost in the worst Nash equilibrium to the minimum social cost possible.

[3] Rosenthal's potential function shows that congestion games are potential. Monderer and Shapley (1996) proved that every potential game can be represented in a form of a congestion game. Thus, classes of potential games and congestion games coincide. Hence, congestion games are essentially the only class of games for which one can show the existence of pure equilibria with an exact potential function.

the intersection between Rosenthal's and Milchtaich's model. A game in this class is defined by a tuple $\Gamma(N, R, (d_r)_{r\in R})$, where N is a set of n players, R is a set of m resources/strategies (a player's strategy consists of any single resource in R) and d_r is a nonincreasing payoff function associated with resource r. The utility of player i for a profile σ is simply given by $u_i(\sigma) = d_{\sigma_i}(n_{\sigma_i}(\sigma))$. We note that these games are nonsymmetric : Players are restricted to choose only one strategy, but they each have their own utility function. Since the utility of a player derived from selecting a single resource depends only on the number of the players doing the same choice, the specific utility function of this player is simply a mapping: $u_i : R \times \{1, \ldots, n\} \to \mathbb{R}$, $(r, k) \mapsto u_i(r, k)$, where u_i decreases with k.

In the following section, we will use a method which attempts to simplify the analysis of singleton congestion games. This method makes use of a technique, initially introduced by Sbabou et al (2010, 2011), concerning the use of the ordinal representation of preferences. Indeed, in the case of nonsymmetric singleton congestion games, we replace the values of the payment functions by their ranks in a preference ordering representing the specific utility function of each player. More formally, a singleton congestion game with player-specific will be represented by a tuple $\Gamma(N, R, \precsim_i)$ where N is a set of n players, R a set of resources and $\precsim_i$ a weak ordering on $R \times \{1, \ldots, n\}$. In the ordinal context, a strategy profile σ^* is a Nash equilibrium of the game Γ if $\sigma^* \succsim_i (\sigma_i, \sigma^*_{-i})$ for all σ_i in R. A congestion vector $\sigma^* = (n_1, \ldots, n_m)$ corresponds to a Nash equilibrium if, for all r, r' in R with $r \neq r'$, we have $(r, n_r) \succsim_i (r', n_{r'} + 1)$. Thus, no player can benefit from joining a group of players sharing a different resource.

3 Congestion games with exact partition

Before introducing the concept of congestion games with exact partition, we give some decisive definitions.

Definition 1 Let $G(N, R, (\precsim_i)_{i\in N})$ be a singleton congestion game. An n-sequence (or sequence of the n-last terms) derived from a weak ordering $\precsim_i$ is a subset T^i of $R \times N$, such that $|T^i| = n$.

In the sequel, when there is no ambiguity, we denote by T^i both the n-sequence T^i and the order (induced by $\precsim_i$) on T^i.

Remark 1 *If the utility function $\precsim_i$ represents a strict ordering, then there is only one n-sequence for each player, outcome of $\prec_i$.*

Example 1 Let $N = \{1, 2, 3\}$ and $R = \{a, b, c\}$. For simplicity, we will denote the couple (r, k) by rk. Suppose that the ordinal utility function for each player is given by the following strictly decreasing ordering:

$$3a \prec_1 3b \prec_1 2b \prec_1 2a \prec_1 3c \prec_1 a \prec_1 \underbrace{2c \prec_1 c \prec_1 b}.$$

$$3c \prec_2 3b \prec_2 3a \prec_2 2a \prec_2 2c \prec_1 c \prec_2 \underbrace{2b \prec_1 b \prec_1 a}.$$

$$3c \prec_3 3b \prec_3 2b \prec_3 3a \prec_3 2c \prec_3 2a \prec_3 \underbrace{c \prec_1 a \prec_1 b}.$$

By definition 1, the unique 3-sequence for each player i are $T^1 = \{2c, c, b\}$, $T^2 = \{2b, b, a\}$ and $T^3 = \{c, a, b\}$.

Definition 2 Let $G(N, R, (\precsim_i)_{i \in N})$ be a singleton congestion game. A configuration of G is a choice of an (ordered) n-sequence for each player $i \in N$.

Thus, a configuration contains an n-sequence for each player. There are several possible configurations for a game. We can represent a configuration by an array of n rows and n columns: The first line contains T^1, the second contains T^2, and so on.

Remark 2 *When the preference orders are strict, there is only one configuration for each player.*

Definition 3 Let $G(N, R, (\precsim_i)_{i \in N})$ be a singleton congestion game and $(T^1, T^2, \ldots, T^n)$ an (ordered) configuration. For each resource $r \in R$, let $\alpha_i(r) = \max\{p : (r, p) \in T^i\}$, be the maximum number of players that can choose the resource r and $\alpha(r) = \max\{p : |i \in N : (r, p) \in T^i| \geq p\}$ be the greater integer p such that, there exist at least p players verifying the condition $\alpha_i(r) \geq p$. The entity $\alpha(r)$ allows to determine the congestion vector that corresponds to Nash equilibrium.

To get a better idea of the above definitions, let us follow them through an example.

Example 2 Consider $N = \{1, 2, 3, 4, 5, 6\}$ and $R = \{a, b, c\}$. Suppose that for each player i and for each resource r the utility function is strictly decreasing in the number of players choosing r. To apply the definitions 2 and 3, it is not necessary to know the utility functions in their entirety: We have just need to know the order of the n-last terms. Therefore, the sequences of the n last termes are only required. Suppose that they are given by:

$$\ldots 4a \prec_1 2c \prec_1 c \prec_1 3a \prec_1 2a \prec_1 a.$$
$$\ldots 4c \prec_2 2b \prec_2 3c \prec_2 2c \prec_2 b \prec_2 c.$$
$$\ldots b \prec_3 c \prec_3 4a \prec_3 3a \prec_3 2a \prec_3 a\ .$$
$$\ldots 3c \prec_4 2a \prec_4 2c \prec_4 b \prec_4 a \prec_4 c.$$
$$\ldots 2c \prec_5 3a \prec_5 c \prec_5 2a \prec_5 b \prec_5 a.$$
$$\ldots 3a \prec_6 2a \prec_6 a \prec_6 2c \prec_6 c \prec_6 b.$$

Now we can represent the individual preferences T^i by the following table:

T^1	$4a$	$\underline{2c}$	c	$\underline{3a}$	$2a$	a
T^2	$4c$	$2b$	$3c$	$\underline{2c}$	$\underline{b}$	c
T^3	$\underline{b}$	c	$4a$	$\underline{3a}$	$2a$	a
T^4	$3c$	$2a$	$\underline{2c}$	$\underline{b}$	a	c
T^5	$\underline{2c}$	$\underline{3a}$	c	$2a$	$\underline{b}$	a
T^6	$\underline{3a}$	$2a$	a	$\underline{2c}$	c	$\underline{b}$

Table 1: The n-sequence for each player

It is easy to find the integer $\alpha(r)$ for each resource r. For example, for the alternative a, we observe that $4a$ does not satisfy the above definition, since we have only two players (1 and 3) so that $4a$ appears in the sequence of the last 6 terms. However, we have four players that $3a$ appears in the line representing T^i. We have: $max\{p : |i \in N : (a,p) \in T^i| \geq p\} = 3$. Hence, $\alpha(a) = 3$. Similarly, we can see that $\alpha(b) = 1$ and $\alpha(c) = 2$.

Now we are ready to define congestion games with exact partition.

Definition 4 A singleton congestion game $G(N, R, (\precsim_i)_{i \in N})$ is called singleton congestion game with exact partition if there exists a configuration $(T^1, T^2, \ldots, T^n)$ with exact partition, i.e., a configuration that satisfies the condition $\sum_{r=1}^{m} \alpha(r) = n$.

In the example 2, we have $\sum_{r \in R} \alpha(r) = \alpha(a) + \alpha(b) + \alpha(c) = 6 = n$. So, the game is an exact partition.

Remark 3 *We already know by Milchtaich (1996) that any singleton congestion game admits at least one Nash equilibrium. The following result provides a simple proof of Milchtaich's result in the case of games of exact partition. Our proof is straightforward (it is not based on reasoning by induction) and does not involve any mechanism of improvement to obtain a Nash equilibrium: This equilibrium is easily constructible from the table including the sequences of the n last terms.*

4 The result

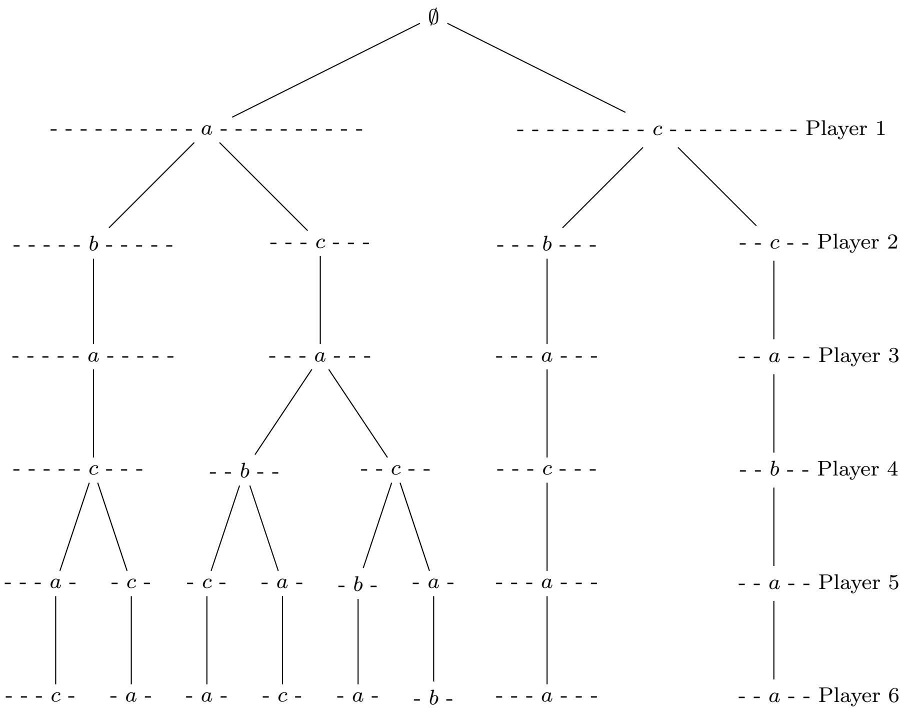
Theorem 1 *Every singleton congestion game satisfying the condition of the exact partition admits at least one Nash equilibrium.*

The following example illustrates the above theorem and shows how to easily build all Nash equilibria in games of exact partition.

Example 3 Let $N = \{1, 2, 3, 4, 5, 6\}$ and $R = \{a, b, c\}$. Individuals' preferences are summarized in Table 1.

By applying our theorem, we can easily find the congestion vector and also all Nash equilibria. We have $\alpha(a) = 3, \alpha(b) = 1$ and $\alpha(c) = 2$, so $\alpha(a) + \alpha(b) + \alpha(c) = 6$, this is a game of exact partition. All Nash equilibria of this game have as congestion vector, the vector $(3a, b, 2c)$. From this vector, we can find all Nash equilibria of the game. We begin by reviewing the classification of pairs $(a, 3), (b, 1)$ and $(c, 2)$ that is to say, $3a$, b and $2c$ in the preferences of each player:
T^1: $b \prec_1 2c \prec_1 3a$ (player 1 may choose a or c as $n_a \succ_1 n_b + 1$, $n_a \succ_1 n_c + 1$, $n_c \succ_1 n_a + 1$ and $n_c \succ_1 n_b + 1$).
T^2: $3a \prec_2 2c \prec_2 b \rightsquigarrow$ player 2 may choose b ou c.
T^3: $2c \prec_3 b \prec_3 3a \rightsquigarrow$ player 3 choose only a.
T^4: $3a \prec_4 2c \prec_4 b \rightsquigarrow$ player 4 may choose b or c.
T^5: $2c \prec_5 3a \prec_5 b \rightsquigarrow$ player 5 may choose b, c or a.
T^6: $3a \prec_6 2c \prec_6 b \rightsquigarrow$ player 6 may choose b, c or a.

We can build the next tree to detect all Nash equilibria:



We can read the above tree as follows: Player 1 has two possible choices: a and c. If player 1 chooses a, player 2 has two possible choices, b or c. If player 2 chooses b, player 3 can only choose a, player 4 has to choose c, player 5 has two possible choices, a or c. If player 5 chooses a, then player 6 must choose only the alternative c (always respecting the number of resources to choose $\alpha(a) = 3$, $\alpha(b) = 1$ and $\alpha(c) = 2$). Thus, the Nash equilibria in this example are: $\sigma_1 = (a, b, a, c, a, c)$, $\sigma_2 = (a, b, a, c, c, a)$, $\sigma_3 = (a, c, a, b, c, a)$, $\sigma_4 = (a, c, a, b, a, c)$, $\sigma_5 = (a, c, a, c, b, a)$, $\sigma_6 = (a, c, a, c, a, b)$, $\sigma_7 = (c, b, a, c, a, a)$, $\sigma_8 = (c, c, a, b, a, a)$.

Proof Let $(T^1, T^2, \dots, T^n)$ be a configuration (ordered) of exacte partition of the game $G(N, R, (\precsim_i)_{i \in N})$. We have :

$$\sum_{r=1}^{m} \alpha(r) = n \tag{4.1}$$

Let $N_0 = N$ and $R_0 = R$. For each $r \in R_0$, consider $B_0(r) = \{i \in N_0 : (r, \alpha(r)) \succsim_i (r', \alpha(r')), \forall r' \in R_0\}$. $B_0(r)$ is the set of players for whom the pair $(r, \alpha(r))$ is preferred (or equivalent) to all other pairs $(r', \alpha(r'))$. Note that when orders are large, several resources r can be such that:

$$(r, \alpha(r)) \succsim_i (r', \alpha(r')), \quad \forall r' \in R_0 \tag{4.2}$$

There exists at least one resource $r \in R_0$ such that: $|B_0(r)| \geq \alpha(r)$ as $\sum_{r=1}^{m} \alpha(r) = n$ (according to (4.1)) and $\sum_{r=1}^{m} |B_0(r)| \geq n$ (the orders T^i not necessarily strict, for the same player, several couples $(r, \alpha(r))$ may be ex-aequo cases verifying (4.2)). Choose (arbitrarily) a resource r_1 in R such that: $|B_0(r_1)| \geq \alpha(r_1)$, let $A_0(r_1)$ be a subset of $B_0(r_1)$ such that: $|A_0(r_1)| = \alpha(r_1)$ and $\forall i \in A_0(r_1)$, $\forall j \in B_0(r_1) \backslash A_0(r_1)$, $\alpha_i(r_1) \geq \alpha_j(r_1)$. We note that:

1. $A_0(r_1)$ is a subset of $B_0(r_1)$ containing $\alpha(r_1)$ players having values $\alpha_i(r_1)$ greater (or equal) than values $\alpha_j(r_1)$ of all the players of $B_0(r_1) \backslash A_0(r_1)$.
2. There may be several possible choices for $A_0(r_1)$.

Once $A_0(r_1)$ constructed, we consider: $N_1 = N_0 \backslash A_0(r_1)$, $R_1 = R_0 \backslash \{r_1\}$. For each $r \in R_1$, let $B_1(r) = \{i \in N_1 : (r, \alpha(r)) \succsim_i (r', \alpha(r')), \forall r' \in R_1\}$. There exists at least one resource $r \in R_1$ such that $B_1(r) \geq \alpha(r)$ because $\sum_{r \in R_1} \alpha(r) = n - \alpha(r_1)$ and $\sum_{r \in R_1} |B_1(r)| \geq n - \alpha(r_1)$. We choose (arbitrarily) $r_2 \in R_1$ such that $B_1(r_2) \geq \alpha(r_2)$. We find a subset $A_1(r_2) = \alpha(r_2)$ and $\forall i \in A_1(r_2)$, for all $j \in B_1(r_2) \backslash A_1(r_2)$, $\alpha_i(r_2) \geq \alpha_j(r_2)$.

Once $A_1(r_2)$ constructed, we have: $N_2 = N_1 \backslash A_1(r_1)$, $R_2 = R_1 \backslash \{r_2\}$ ($= R \backslash \{r_1, r_2\}$). For each $r \in R_2$, let: $B_2(r) = \{i \in N_2 : (r, \alpha(r)) \succsim_i (r', \alpha(r'))$, $\forall r' \in R_2\}$. There exists at least one resource $r \in R_2$ such that $B_2(r) \geq \alpha(r)$. We choose $r_3 \in R_2$ such that $B_2(r_3) \geq \alpha(r_3)$. We find a subset $A_2(r_3)$ of $B_2(r_3)$ such that : $|A_2(r_3)| = \alpha(r_3)$ and $\forall i \in A_2(r_3)$, $\forall j \in B_2(r_3) \backslash A_2(r_3)$, $\alpha_i(r_3) \geq \alpha_j(r_3)$.

In general, supposing $A_{k-1}(r_k)$ built (for $k \geq 1$) there are two cases:

1. $|A_0(r_1)| + \ldots + |A_{k-1}(r_k)| = n$, in this case we stop the process.
2. $|A_0(r_1)| + \ldots + |A_{k-1}(r_k)| < n$, in this case we continue the process by considering: $N_k = N_{k-1} \backslash A_{k-1}(r_k)$, $R_k = R_{k-1} \backslash \{r_k\}$. For each $r \in R_k$, let $B_k(r) = \{i \in N_k : (r, \alpha(r)) \succsim_i (r', \alpha(r')), \forall r' \in R_k\}$. There exists at least one

resource $r \in R_k$ such that $B_k(r) \geq \alpha(r)$ as $\sum_{r \in R_k} \alpha(r) = n-(\alpha(r_1)+\ldots+\alpha(r_k))$ and $\sum_{r \in R_k} |B_k(r)| \geq n-(\alpha(r_1)+\ldots+\alpha(r_k))$. We choose (arbitrarily) $r_{k+1} \in R_k$ such that $B_k(r_{k+1}) \geq \alpha(r_{k+1})$. We extract a subset $A_k(r_{k+1})$ of $B_k(r_{k+1})$ such that: $|A_k(r_{k+1})| = \alpha(r_{k+1})$ and $\forall i \in A_k(r_{k+1})$, $\forall j \in B_k(r_{k+1}) \backslash A_k(r_{k+1})$, $\alpha_i(r_{k+1}) \geq \alpha_j(r_{k+1})$.

Note that when the process stops at the step l, we have : The profile $\sigma^* = (\sigma_1^*, \ldots, \sigma_n^*)$ is a Nash equilibrium, with $\sigma_i^* = r_{k+1}$ if and only if $i \in A_k(r_{k+1})$, for all i in N.

For all $i \in A_0(r_1)$, $\sigma_i^* = r_1$ and by definition of $B_0(r_1)$ we get $(r_1, \alpha(r_1)) \succsim_i (r', \alpha(r')), \forall r' \in R_0 = R$ (i.e., $(r', \alpha(r')) \succsim_i (r', \alpha(r')+1)$). Thus, $(r_1, \alpha(r_1)) \succsim_i (r', \alpha(r')+1)$ $\forall r' \in R$. For all $i \in A_1(r_2)$, $\sigma_i^* = r_2$ and by definition of $B_1(r_2)$:

$$(r_2, \alpha(r_2)) \succsim_i (r', \alpha(r')), \quad \forall r' \in R_1 = R \backslash \{r_1\} \tag{4.3}$$

Thus, $(r_2, \alpha(r_2)) \succsim_i (r', \alpha(r')+1)$ $\forall r' \in R \backslash \{r_1\}$. Remains to show that: $(r_2, \alpha(r_2)) \succsim_i (r_1, \alpha(r_1)+1)$. Suppose the contrary. That is to say:

$$(r_1, \alpha(r_1)+1) \succ_i (r_2, \alpha(r_2)) \tag{4.4}$$

We will have $(r_1, \alpha(r_1)) \succ_i (r_2, \alpha(r_2))$ and using (4.3), we obtain : $(r_1, \alpha(r_1)) \succsim_i (r', \alpha(r'))$, $\forall r' \in R_0 = R$. So, i would be in $B_0(r_1)$. Accordingly to (4.4), $(r_1, \alpha(r_1)+1) \succ_i (r_2, \alpha(r_2))$, always using (4.3), we will have: $(r_1, \alpha(r_1)+1)$ appears into T^i (because $(r_2, \alpha(r_2))$ appears into T^i). i was not retained into $A_0(r_1)$. It follows that all players j of $A_0(r_1)$ are such that $(r_1, \alpha(r_1)+1)$ appears into T^j. This is absurd because otherwise $\alpha(r_1) \geq \alpha(r_1)+1$ (that is to say that there will be at least $\alpha(r_1)+1$ players j having $(r_1, \alpha(r_1)+1)$ in T^j), which is impossible. So, $(r_2, \alpha(r_2)) \succsim_i (r_1, \alpha(r_1)+1)$. Therefore, we have, for all $i \in A_1(r_2)$, $(r_2, \alpha(r_2)) \succeq_i (r', \alpha(r')+1)$, $\forall r' \in R$.

In general, for all $i \in A_k(r_{k+1})$, we have by definition of $B_k(r_{k+1})$: $(r_{k+1}, \alpha(r_{k+1})) \succsim_i (r', \alpha(r')), \forall r' \in R_k$, with $R_k = R \backslash \{r_1, \ldots, r_k\}$. So $(r_{k+1}, \alpha(r_{k+1})) \succsim_i (r', \alpha(r')+1)$, $\forall r' \in R \backslash \{r_1, \ldots, r_k\}$. A reasoning similar to that concerning $\sigma_i' = r_2$, shows that: $(r_{k+1}, \alpha(r_{k+1})) \succsim_i (r', \alpha(r')+1)$, $\forall r' \in \{r_1, \ldots, r_k\}$. Finally, for all $i \in A_k(r_{k+1})$, $(r_{k+1}, \alpha(r_{k+1})) \succsim_i (r', \alpha(r')+1)$, $\forall r' \in R$. This, shows that σ^* is a Nash equilibrium. $\square$

5 Conclusion and open problems

Our contribution was obtained by reviewing the results of Milchtaich for non-symmetric singleton congestion games. The study of this kind of games, led us to isolate a sub-class of congestion games that we have called "congestion games with exact partition", for which we have shown the ability to find all Nash equilibria. We presented an alternative proof to the one given by Milchtaich, to show how to calculate these equilibria more easily. However, our method is not valid for finding Nash equilibria in congestion games without exact partition. In the latter case, we need to slightly modify the method described above, in order to obtain a congestion vector corresponding to a Nash equilibrium, which will be the aim of our future research.

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